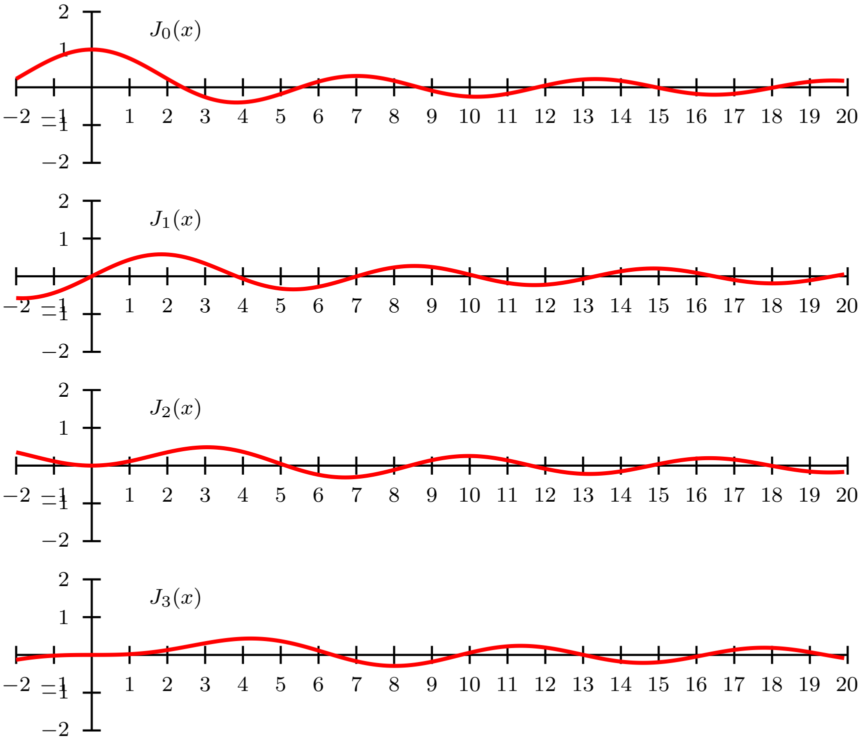
# Circular membrane

**Wave equation for a disc** $0 < r < R$**:** $u_{tt} = c^2 \nabla^2 u = c^2 \left[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right]$, $u(R, \theta, t) = 0$.

**Inner product:** Since $dx\,dy = r\,dr\,d\theta$, we have $u \cdot v = \int_{-\pi}^{\pi} \int_0^R u(r,\theta)\,\overline{v}(r,\theta)\,r\,dr\,d\theta$

**Separation:** If $u(r, \theta, t) = W(r)Q(\theta)G(t)$, then $r^2 W_{rr} + rW_r + (k^2 r^2 - n^2)W = 0$, $Q_{\theta\theta} + n^2 Q = 0$, and $G_{tt} + c^2 k^2 G = 0$. For $Q$ to be continuous, $n$ must be an integer and $W(r) \sim J_n(kr)$, where $J_n$ is the $n$-th Bessel function. [1]




Let $\alpha_{mn}$ be the positive zeros of $J_n$. In order for $J_n(kR) = 0$, we must have $k = k_{mn} = \frac{\alpha_{mn}}{R}$.

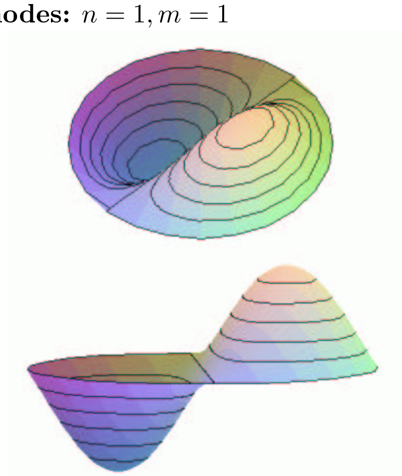
**General solution:**

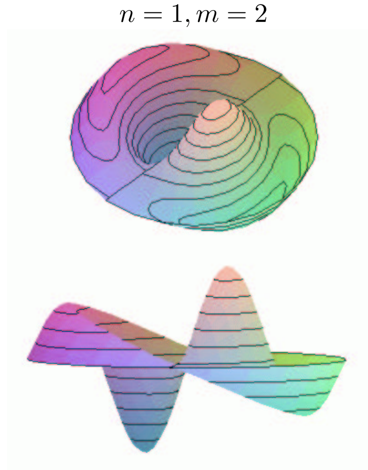
$$u = \sum_{n=0}^{\infty} \left[ \cos(n\theta) \sum_{m=1}^{\infty} J_n(k_{mn}r) \left[ A_{mn} \cos(ck_{mn}t) + B_{mn} \sin(ck_{mn}t) \right] + \sin(n\theta) \sum_{m=1}^{\infty} J_n(k_{mn}r) \left[ A^*_{mn} \cos(ck_{mn}t) + B^*_{mn} \sin(ck_{mn}t) \right] \right]$$

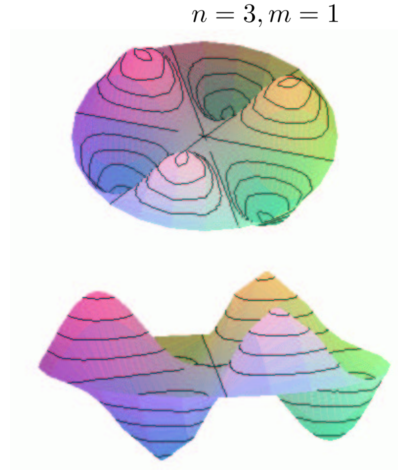
**Initial data example:** If $u_t(r, \theta, 0) = 0$ and $u(r, \theta, 0) = f(r, \theta)$ is an even function of $\theta$, then

$$u = \sum_{n=0}^{\infty} \cos(n\theta) \sum_{m=1}^{\infty} A^*_{mn} J_n(k_{mn}r) \cos(ck_{mn}t), \text{ where } A^*_{mn} = \frac{2}{\pi R^2 J_{n+1}(\alpha_{mn})} \int_{-\pi}^{\pi} \int_0^R f(r,\theta)\,\cos(n\theta)\,J_n\left(\frac{\alpha_{mn}}{R} r\right)\,r\,dr\,d\theta$$

**Eigenmodes:** $n = 1, m = 1$ $\quad$ $n = 1, m = 2$ $\quad$ $n = 3, m = 1$









[1] Friedrich Bessel (1784–1846)