Limits *by Dr. Dmitry Gokhman* 1996 .... *Take it to the limit one more time* — The Eagles

Definition: $L = \lim_{x \to x_0} f(x)$ means $f(x)$ is as close to L as we want, as long as we choose x close enough to x_0.
In other words, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Definition: $f(x)$ is <u>continuous</u> at x_0 means $\lim_{x \to x_0} f(x) = f(x_0)$.

Here are some rules (assuming the constituent limits exist): $\boxed{u(x) \le v(x) \Rightarrow \lim_{x \to x_0} u(x) \le \lim_{x \to x_0} v(x)}$

$$\boxed{\begin{array}{l} \lim_{x \to x_0} c = c \\ \lim_{x \to x_0} x = x_0 \end{array}} \quad \boxed{\begin{array}{l} \lim_{x \to x_0} (u(x) + v(x)) = \lim_{x \to x_0} u(x) + \lim_{x \to x_0} v(x) \\ \lim_{x \to x_0} (u(x) \cdot v(x)) = \lim_{x \to x_0} u(x) \cdot \lim_{x \to x_0} v(x) \end{array}} \quad \boxed{\begin{array}{l} u \text{ is continuous at } x_0 \Rightarrow \\ \lim_{x \to x_0} (u(f(x)) = u\left(\lim_{x \to x_0} f(x)\right) \end{array}}$$

Sandwich Rule (Squeeze Law): If $u(x) \le f(x) \le g(x)$ and $\lim_{x \to x_0} u(x) = \lim_{x \to x_0} v(x) = L$, then $\lim_{x \to x_0} f(x) = L$.

Intermediate and Extreme Value Theorems: If f is continuous on $[a, b]$,
then f attains a maximum and a minimum value on $[a, b]$ as well as all intermediate values.

Differentiation *by Dr. Dmitry Gokhman* 1996 ...... *du means "a little bit of" u* — Silvanus P. Thompson (1910)

Geometrical Definition: $f'(x_0)$ is the slope of the line tangent to the graph of $y = f(x)$ at $(x_0, f(x_0))$.

Analytical Definition: $f'(x_0)$ is approximated by the slope of a secant line $f'(x_0) \approx \dfrac{f(x) - f(x_0)}{x - x_0}$.

The approximation gets better as x approaches x_0, so $\boxed{f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}}$

Linear Approximation: Given a function $f(x)$ we want to approximate it with a line $\ell(x) = m \cdot x + b$.
The line should agree with f at one point x_0, i.e. $f(x_0) = m \cdot x_0 + b$. Thus, $b = f(x_0) - m \cdot x_0$, so $\ell(x) - f(x_0) = m \cdot (x - x_0)$.
We want $f(x) \approx \ell(x)$ when x is close to x_0. This happens when $\ell(x)$ is the tangent line at $(x_0, f(x_0))$, so $m = f'(x_0)$.

Another way to see this is to consider the error $\varepsilon(x) = \ell(x) - f(x)$. We require $\lim_{x \to x_0} \dfrac{\varepsilon(x)}{x - x_0} = 0$

If $\ell(x)$ exists satisfying this requirement, we say that $f(x)$ is <u>differentiable</u> at x_0.

Since $\varepsilon(x) = m \cdot (x - x_0) + f(x_0) - f(x)$, we have $\lim_{x \to x_0} \left(m - \dfrac{f(x) - f(x_0)}{x - x_0} \right) = 0$, Thus, $m = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$.

The Differential: Let $\Delta x = x - x_0$ and $\Delta f = f(x) - f(x_0)$.
Since $f(x) \approx \ell(x)$, $\Delta f = f(x) - f(x_0) \approx \ell(x) - f(x_0)$. We define the <u>differential</u> $df = \ell(x) - f(x_0)$.
We obtain $\boxed{\Delta f \approx df}$ and $\boxed{df = f'(x_0)\Delta x}$

If f is already linear, then our approximation is exact, i.e. $df = \Delta f$ and f' is the slope of f.

In particular, $\boxed{dc = 0, \quad c' = 0}$ $\boxed{dx = \Delta x,, \quad x' = 1}$ This is the basis for the notation: $\boxed{df = f'dx}$ or $\boxed{f' = \dfrac{df}{dx}}$

Interaction of differentiation with operations on functions:

Sum	$d(u + v) = du + dv$	$(u + v)' = u' + v'$
Product	$d(u \cdot v) = du \cdot v + u \cdot dv$	$(u \cdot v)' = u' \cdot v + u \cdot v'$
Chain	$du = d(u(v)) = \dfrac{du}{dv} \cdot dv$	$(u(v))' = u'(v) \cdot v'$

Note that differentiation is <u>linear</u>, i.e.
$d(a \cdot u(x) + b \cdot v(x)) = a \cdot du + b \cdot dv$,
$(a \cdot u(x) + b \cdot v(x))' = a \cdot u' + b \cdot v'$

Rules of differentiation for a few specific functions:

Power	$d(x^a) = a \cdot x^{a-1} \cdot dx$	$(x^a)' = a \cdot x^{a-1}$
Exp	$d(a^x) = \ln a \cdot a^x \cdot dx$	$(a^x)' = \ln a \cdot a^x$
Log	$d(\log_a x) = (\ln a \cdot x)^{-1} \cdot dx$	$(\log_a x)' = (\ln a \cdot x)^{-1}$

$d(\sin x) = \cos x \cdot dx$	$d(\cos x) = -\sin x \cdot dx$
$d(\tan x) = \sec^2 x \cdot dx$	$d(\cot x) = -\csc^2 x \cdot dx$
$d(\sec x) = \tan x \cdot \sec x \cdot dx$	$d(\csc x) = -\cot x \cdot \csc x \cdot dx$

Example: Suppose $f(x) = x^2$. Then $f(x) = f(x - x_0 + x_0) = f(dx + x_0) = (x_0 + dx)^2 = x_0^2 + 2x_0 \cdot dx + (dx)^2$.
Note that if dx is small, then $(dx)^2$ is a lot smaller. The linear approximation to $f(x)$ is $\ell(x) = x_0^2 + 2x_0 \cdot dx$.
Thus, $df = 2x_0 \cdot dx$ and $f'(x_0) = 2x_0$. Compare this to the power rule.

Mean Value Theorem: If f is differentiable on an interval (a, b) and, in addition, continuous at the end-points,
then there exists c such that $a < c < b$ and $f(b) - f(a) = f'(c)(b - a)$, i.e. $\boxed{f'(c) = \dfrac{f(b) - f(a)}{b - a}}$

Integration

.... *The integral simply means "the whole" — $\int$ is merely a long S* — Silvanus P. Thompson (1910)

Integration is the process of reconstructing functions from their "little bits" — the differentials.
Suppose we have a differential $g(x) \cdot dx$ given by a continuous function $g(x)$ on an interval $[a, b]$.
We want to find a function $f(x)$ such that $df = g \cdot dx$, i.e. $f' = g$. This is written $f(x) = \int g(x)\,dx$.
We call f an indefinite integral (anti-derivative) of g. Clearly f is not unique as it may have an arbitrary constant added.

Example: If $g(x)$ is particularly simple, e.g. a constant $g(x) = m$, we can guess easily: $f(x) = m \cdot x + c$.
If we know $f(x_0)$, we can determine $c = f(x_0) - m \cdot x_0$, so $f(x) = f(x_0) + m \cdot (x - x_0)$.

For slightly more complicated $g(x)$ we can use integration rules based on differentiation rules.

Linearity	$\int (a \cdot u(t) + b \cdot v(t))\,dt = a \cdot \int u(t)\,dt + b \cdot \int v(t)\,dt$
Substitution (chain)	$\int u(v(t)) \cdot v'(t)\,dt = \int u(v)\,dv$
By parts (product)	$\int u \cdot dv = u \cdot v - \int v\,du$

$\int t^n dt = t^{n+1}/(n+1) + c, \quad n \neq -1$
$\int t^{-1} dt = \ln t + c$

Geometrical Construction: If $g(x) > 0$, define $f(b)$ to be the area between the graph of $y = g(x)$, the x axis and the vertical lines $x = a$ and $x = b$. The areas where $g(x) < 0$ are subtracted. Varying b (letting $b = x$) we write $f(x) = \int_a^x g(t)\,dt$.
The change in area as we increase x by a small dx is nearly rectangular with base dx and height $g(x)$. Thus, $df = g(x)\,dx$.

Additivity: $\boxed{\int_a^b g(t)\,dt = \int_a^c g(t)\,dt + \int_c^b g(t)\,dt}$ with the convention that $\int_b^a g(t)\,dt = -\int_a^b g(t)\,dt$.

Order Preservation: $\boxed{u(x) \leq v(x) \Rightarrow \int_a^b u(t)\,dt \leq \int_a^b v(t)\,dt}$

Approximation: For x near x_0 we can estimate $f(x)$, by approximating $g(x)$ with a constant, say $g(x_0)$.
Then as in the example $f(x) \approx f(x_0) + g(x_0) \cdot (x - x_0)$.
This corresponds to linear approximation of $f(x)$ at x_0, where $f(x) - f(x_0) \approx f'(x_0) \cdot (x - x_0)$.

Analytical Construction: A large interval must be partitioned into smaller sub-intervals.
This corresponds to approximating $g(x)$ by a step function.
A partition is determined by a finite set of points in the interval: $P = \{a \leq a_1 \leq a_2 \leq ... \leq a_{n-1} \leq b\}$.
On each subinterval (a_i, a_{i+1}) we substitute the end-points into the approximation formula: $f(a_{i+1}) - f(a_i) \approx g(a_i)(a_{i+1} - a_i)$.
Riemann says: g can be evaluated at any point x_i in the interval (a_i, a_{i+1}), so $\boxed{f(a_{i+1}) - f(a_i) \approx g(x_i)(a_{i+1} - a_i)}$
The right side is a small approximately rectangular bit of area under the graph of $y = g(x)$.
We sum the above formula over all the intervals (letting $a_0 = a$, $a_n = b$ and $\Delta a_i = a_{i+1} - a_i$ for convenience)
$f(a_1) - f(a_0) + f(a_2) - f(a_1) + ... + f(a_n) - f(a_{n-1}) \approx g(x_0)\Delta a_0 + g(x_1)\Delta a_1 + ... + g(x_{n-1})\Delta a_{n-1}$

The left side telescopes to $f(a_n) - f(a_0)$, so $\boxed{f(b) - f(a) \approx \sum_{i=0}^{n-1} g(x_i)\Delta a_i}$ The right side is called a Riemann sum.

Refine the partition (add points) such that the maximum sub-interval length (written $|P|$) decreases.
The right side gives a better approximation to the "area" $\int_a^b g(x)\,dx$ (the definite integral of $g(x)$ from a to b).

Take limit: $\boxed{f(b) - f(a) = \lim_{|P| \to 0} \sum_{i=0}^{n-1} g(x_i)\Delta a_i = \int_a^b g(x)\,dx}$ Vary b (let $b = x$): $\boxed{f(x) = f(a) + \int_a^x g(t)\,dt}$

Fundamental Theorem of Calculus: We can rephrase the above completely in terms of g alone or f alone.
We see that differentiation (slope) and integration (area) are inverse operations in the following sense:

Barrow's Rule: $\boxed{\int_a^b df = f(b) - f(a), \quad \int_a^b f'(x)\,dx = f(b) - f(a)}$ and $\boxed{d\left(\int_a^x g(t)\,dt\right) = g(x)\,dx, \quad \left(\int_a^x g(t)\,dt\right)' = g(x)}$