Newton's binomial and Pascal's triangle

Playing lotto: You have n numbered balls and k balls are picked out in sequence ($0 \le k \le n$). When picking the first ball you have n choices. For the next ball $n-1$ choices remain, and so on. Thus, the number of different ways this can be done, denoted by A^n_k, is $n(n-1)...(n-k+1)$. In the special case when $k=n$ we have $A^n_n = n(n-1)...1$, which is denoted by $n!$ (n factorial). This is the number of ways n balls can be reordered (permuted). Note that since there is only one way permute no balls, $0! = 1$. With this notation we have

$$A^n_k = n(n-1)...(n-k+1) = \prod_{i=0}^{k-1}(n-i) = \frac{n!}{(n-k)!}$$

Making choices: Despite the fact that the balls are picked out in sequence, their order does not affect your success at lotto. We may consider any sequences of length k consisting of the *same* set of balls as equivalent. The number of such sequences is $A^k_k = k!$. Thus, the number of truly different selections, denoted by C^n_k, is

$$C^n_k = \frac{A^n_k}{A^k_k} = \frac{n(n-1)...(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Splitting choices: Factorials get huge pretty quickly, so taking their ratios is not the best way to compute. One way to ease the computation is to split the choice. If you choose k balls out of n you will either pick the first one or not. If you pick the first ball, you still have to choose $k-1$ more out the remaining $n-1$ balls. If you don't pick the first ball, you must pick all k balls out of the remaining $n-1$. Thus,

$$C^n_k = C^{n-1}_{k-1} + C^{n-1}_k$$

This idea of recursion is behind the Pascal triangle construction which offers fast computation of C^n_k. Here are the first few rows of Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Thus, for example $C^3_1 = 3$ and $C^4_2 = 6$. Note how each entry is the sum of the entries above it. Also note the obvious symmetry $C^n_k = C^n_{n-k}$.

Foiling: To raise a binomial $a+b$ to the n-th power let us write it as a product of n factors $(a+b)^n = (a+b)(a+b)...(a+b)$ and use the distributive law to multiply things out (this is called foiling). The result will be a sum of products of a's and b's with the total power n. Let us collect terms with same powers of a (the power of b must be n minus the power of a). If we choose a in each factor $a+b$ we obtain a^n. If we replace one of the a's with b, we can take that b out of any of the n factors $a+b$, so there will be n terms $a^{n-1}b$. In general, if we replace k a's with b's, we will have C^n_k terms $a^{n-k}b^k$. Thus,

$$(a+b)^n = a^n + na^{n-1}b + C^n_2 a^{n-2}b^2 + ... + b^n = \sum_{k=0}^{n} C^n_k a^{n-k} b^k$$

For example, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.