20 Research, Ancient and Modern

Chapter 1. On government research establishments

THE main justification for mathematical research, and I think Professor Coulson* would have said the same, is that it is one of the oldest and most splendid endeavours of mankind. And that could well be the end of my talk.

But whether or not the government can afford to support vast numbers of us on fat salaries, to do what we enjoy, is another matter. The problem is as old as history. Recall that apocryphal story about Euclid told by Stobaeus[1]:

"Someone who had begun to read geometry with Euclid, when he had learnt the first theorem asked Euclid 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, since he must make gain out of what he learns'."

Now you may laugh at Euclid's apparent gentle sarcasm, but I am not so sure. Judging from the humourlessness of Euclid's mathematical style, and remembering his position as a head of department of a new government research establishment, and recalling Professor Bondi's words about *his* experiences in a similar position, one could easily interpret Euclid's reply at face value. Anyway why did he offer threepence, when a penny would have made the point just as well? He probably had a research budget of £30, and being commissioned to produce a mammoth standard reference work on mathematics had, with an administrator's acumen, estimated it at about 1000 propositions, and was merely making use, like Bondi, of masses of cheap research student labour, as opposed to a few expensive professors. And that, alas, was probably the beginning of the bad effect of paying for, and promoting because of, research.

You may ask why do I describe the famous Mouseion at Alexandria as a government research establishment. In fact it was probably the first,[2] and it probably possessed all the attributes of a research establishment, tenured posts, excellent library and plenty of slave labour. One can easily conjecture the kind of conversation that must have once taken place.

One evening Alexander the Great as a youth comes up to his tutor and says:

Alexander: "I have a problem."

Aristotle (*who happened to be his tutor*): "Yes?"

Alexander: "In my plan to conquer the world it is obviously best to use a single well organised army. But as I capture each country, and then move on to the next, how do I keep control of the previous country?"

Aristotle (*after a pause, with a far seeing glint in his eye*): "Aha! I think I have the solution. You want to found a government research establishment. You could even name it after yourself. Then the sociology department (reference 2, p. 20) could manufacture suitable religions grafted on to the appropriate local beliefs that would keep the natives happy."

"As a matter of fact," and at this juncture Aristotle's tone of voice becomes noticeably casual, "as a matter of fact I have a very good student* who could do the architecture for you—he's eager to experiment with white marble—and another senior student† who would make a splendid first director of the place."

Aristotle's voice regains its normal timbre

"I suppose you'll have to have an arts man as first librarian—and there is an elderly Homer scholar‡ who would do—and he would have the advantage of being near retiring age so that as soon as he'd done the chore of setting up the catalogue system you could get rid of him and replace him by a proper scientist."

Aristotle's voice goes casual again. "And as a matter of fact I have just the man§ for the job, a student who is a brilliant all-rounder, interested in astronomy, geography, literature, the lot, but he needs a few more years of research before he takes on administrative chores. Oh yes—and I have another young student who's a bit of a crank, but marvellous with his hands. His ambition is to build a giant lighthouse, but he can't get any funds. But in a government research establishment this would be well worth the cost, just from the prestige point of view alone, besides being actually quite a useful piece of equipment."

"I suppose you'll have to have a philosophy department, although to tell the truth the subject is a bit played out after Plato and myself, and most of my current students are rather second rate (reference 2, p. 40). On the other hand biology, psychology and medicine are really up and coming new subjects, and I have a splendid young man¶ who has done some fascinating work on the psychology of sex and nervous breakdowns, who would be ideal to head a research group."

"And let me see—you'll need a mathematician of course—and although I don't have any suitable students of my own available just at this moment, there is a young man** in Plato's academy. Not that he's very good at research, in fact I doubt he'll ever make his PhD, but he's quite a good scholar, and quite good at editing things. And although he's a bit humourless, he would make an excellent administrator, and so I'd recommend hiring him to set up the mathematics department."

"Oh—and another point—if I were you I would choose somewhere on the mediterranean coast, with a nice climate and a sandy beach with good bathing facilities, and not too far from the main shipping lanes.

* Professor Coulson would have been giving this talk, but for his untimely death.

* Dinocrates.
† Demetrius Phalerus.
‡ Zenodotus.
§ Eratosthenes.
‖ Sostratus.
¶ Erasistratus.
** Euclid.

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E. C. Zeeman, Catastrophe Theory: Selected Papers, 1972-1977. Published by Addison-Wesley Publishing Company, Inc. Advanced Book Program, Reading, Massachusetts, 01867.

As a matter of fact I had a vacation last year at just such a place, a little island called Ras-el-Tin (reference2, p.7). For that way you'll not only be able to attract some decent academics on to the staff, but you'll also guarantee a good flow of visitors each summer to keep the place academically alive. In fact it might even last a few centuries."

And that's exactly what Alexander did, in every detail, when he was 23.

Previous to Alexandria, there had only been private academies, which lacked permanence because they depended upon personalities, and tended to dissolve when the latter moved or died. The most famous example was Plato's Academy in Athens. Plato himself must have been one of the best, and stupidest, research supervisors in mathematics of all time. According to Tzetzes[3] he even had a notice written up over the porch of his academy: "Let no one unversed in geometry enter my doors," but when you come to see what he himself thought geometry was, you find a rather pedantic insistence upon ruler and compass constructions, which was stupid. It is a regrettable tendency amongst some mystics and philosophers to become so obsessed with some small facet of mathematics or science that happens to capture their imagination, that it biases their view of the whole.

The trouble was that Plato had so strong a personality, he almost managed to redirect the whole steam-roller of Greek mathematics in the wrong direction (as Cauchy later did succeed in doing, by discovering Cauchy's Theorem, and thereby redirecting the whole of nineteenth century mathematics away from real analysis into the jaws of complex analysis).

Chapter 2. Rediscovering Eudoxus of Cnidus

Luckily for the Greeks, Plato had at least one research student who was his match, and who was, in my opinion, the greatest of all the Greek mathematicians, Eudoxus. He was born in Cnidus about 408 BC, and Plato was about 20 years his senior (see reference 3 vol.1, pp. 320–334). Eudoxus was 23 when he became Plato's student, and so Plato must have been in his early forties, approaching the height of his powers. Meanwhile Cnidus was in what is now mainland Turkey, and so I suppose you could call Eudoxus something of a young Turk. One can imagine the opening conversation:

"Well young man, I have here 4 problems which you might like to try your hand at, left here by a fellow called Zeno."

One only has to look at Plato's Dialogues to know that he always went straight to the heart of the matter irrespective of whom he was talking to. And I believe this is a great virtue in a research supervisor: myself I much prefer Professor Atiyah's gold to Professor Roger's tin, especially during the first year of research.

But imagine Plato's astonishment when Eudoxus returns shortly with a closely written sheaf of papyrus claiming to have solved the lot. Then Plato's second virtue as a research supervisor comes out: insistence on clarity of communication.

"My dear young man," he says handing it straight back to the crestfallen Eudoxus, "you must *explain* the solution to me in words of one syllable, just as I explained the problem to you. We philosophers believe in the value of debate."

And as a matter of fact this used to be exactly Norman Steenrod's description of how to write a mathematical paper: "imagine you are going on a long walk with a friend, and you are telling him about the theorem—write the paper in *that* order."

But to return to Eudoxus' predicament of having to do mathematical research in an academy of philosophy. In effect he had to reduce the proof to as short a time as those argumentative philosophers would allow him to hold the floor. And being the greatest of all the Greek mathematicians, he meets this challenge. In effect he reduces the proof to one line, Definition 5 in Euclid Book V (see reference 1, vol. 2, p. 114, 120–129).

The problem was to define *ratio* between incommensurable magnitudes, when there was no definition of real numbers, nor any definition of how to add or multiply irrationals. His solution was to define the correct equivalence relation between pairs as follows. Let N denote the positive integers.

Eudoxus' Definition of Equivalence

$(a, b) \sim (a', b')$ if, for all $m, n \in N$, $ma \gtreqless nb$ as $ma' \gtreqless nb'$. The equivalence class of the pair (a, b) is called the *ratio* (λόγος) and is denoted $a{:}b$. Thus

$$a{:}b = a'{:}b'.$$

Now the beauty of the definition is its generality, because a, b, a', b' can be any kind of magnitudes, space intervals, time intervals, areas, volumes, musical notes, integers, rationals, irrationals, etc.—in fact the elements of any ordered set on which N operates. In the special case that they are real numbers (which *we* know to exist by Weierstrass and Dedekind, but which the Greeks did not), we can divide, and so the condition reduces to

$$\frac{a}{b} \gtreqless \frac{n}{m} \text{ as } \frac{a'}{b'} \gtreqless \frac{n}{m}.$$

Therefore $a/b = a'/b'$ because they determine the same Dedekind cut of the rationals.[4] Hence we can identify $a{:}b = a/b$. But Eudoxus' definition is much more general* than merely referring to the reals—it is the beginning of abstract algebra, and I shall develop arguments to suggest that he was by far the greatest algebraist of the Greeks, as well as being by far their greatest analyst, and in the very top rank both as geometer and astronomer.

Observe, in passing, that one can trace a direct line from Eudoxus to the discovery of the reals.

Eudoxus–Euclid–Bolzano–Weierstrass–Dedekind. The key link in this case is Bolzano†. He in his autobiography[5] confesses that the one work that really switched him on was the beauty of Euclid Book V. It was Bolzano who first introduced the ϵ, δ technique, and gave a rigorous definition of continuity. Bolzano never uses infinitesimals, while Cauchy at the same time was still using them freely. I think one can trace Bolzano's inspiration for this technique directly to Euclid Book V.

Now according to Proclus[1,3] Euclid Book V summarised Eudoxus' theory of proportion, which was traditionally recognised by the Greeks as being the "crown of Greek

* According to Proclus[3]: "Eudoxus . . . was the first to increase the number of so called general theorems."

† I am indebted to David Fowler for not only introducing me to the early nineteenth century, but also for a great deal of enjoyment of mathematics.

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mathematics."[6] Personally, although I too admire the beauty of the book, and its rigour, and recognise it as the earliest surviving work on modern abstract algebra, nevertheless I suspect that it is a travesty. I suspect that Euclid ruined Eudoxus' theory by first misunderstanding it and then reporting only a fragment of it in the wrong order, so that he may have successfully prevented it from even being fully rediscovered. Because, alas, no actual words of Eudoxus survive today—the originals were probably burnt when the library at Alexandria was destroyed by Theophilus in 391 AD (reference 2, p. 55).

But first let me explain how Euclid used the little bit of Eudoxus that he did report. Before Eudoxus the Greeks were unable to state rigorously any similarity theorems. For example in Fig. 1 the equality $a:b = a':b'$ can only be stated either if one has the real numbers and a definition of division (neither of which the Greeks had) or if one has Eudoxus' definition of ratio. That is why Euclid had to postpone all similarity theorems to Book VI after Book V.

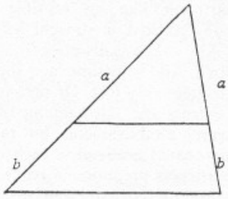

Fig. 1

That is also why Professor Penrose, as a boy, was astonished to discover the delightful similarity proof of Pythagoras' theorem:

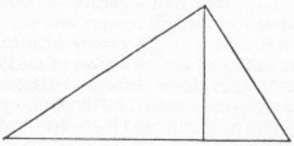

Fig. 2

For thanks to the dreadful influence of Euclid's pedantry upon schoolmasters for 2000 years this particular jewel of a proof was officially suppressed, because, in the interests of rigour*, it should properly be postponed until after either the proof of the existence of the real numbers, or Euclid Book V, neither of which is accessible to school mathematics. As a result school children were denied the benefit of intuition of the reals and forced to swallow the 47 propositions of Euclid Book I, in order to reach Euclid's ingenious but lumpish proof, independently of the reals. And of course what they gained on the swings they lost on the roundabouts, because they also had to swallow the rather shady axiom system of Book I.

Today we get exactly the same phenomenon, but *worse*. Because, in the interest of rigour*, and thanks to Euclid's modern counterpart Bourbaki, poor young French children have the axiom system for the reals forced down their unwilling gullets.

* Rigor mortis.

Myself I prefer the beautiful Chinese proof of Pythagoras' Theorem:

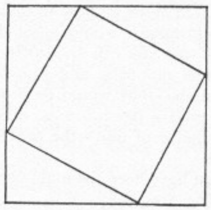

Fig. 3

For this has not only a pleasing symmetry, but the superior advantage of being provable either way, either with the reals, or by the chopping up methods of Euclid Book I.

Another essential place where Euclid has to use Eudoxus' definition of ratio is in Book XII. Here $a:b$ is the ratio of the volumes of two cones of the same height, and $a':b'$ is the ratio of the areas of their bases. This time the proof $a:b = a':b$ (again without the reals and hence without integral calculus) is a real *tour de force*, also due to Eudoxus, for which he invented the theory of exhaustion (assuming false, and proving a contradiction by a finite inductive chopping up process). From this he deduces:

$$\text{volume of cone} = \tfrac{1}{3}\ \text{base} \times \text{height}.$$

There was no other rigorous proof of this simple fact until Dedekind[4] discovered the reals in 1854, thereby endorsing the use of integral calculus. And eventually in 1900 Dehn's solution[7] of Hilbert's 3rd problem[8] showed that, without the reals, this was the only way Eudoxus could have done it. This is an example of Eudoxus the analyst—not the only example because for instance he also invented the hippopede to describe planetary motion.[9]

But I am more interested in Eudoxus the algebraist, and so now let us turn to the more serious crimes of Euclid, the crimes of omission rather than commission. Euclid is the classic example of the dangers of putting a "good scholar" in charge of a university, or research institute, or curriculum reform, as opposed to the "researcher." *For good scholars tend to compartmentalise knowledge, while researchers try to synthesise it.* So Euclid put pure mathematics in the mathematics department, and applied mathematics in the applied departments, in the astronomy and geography departments at Alexandria, insisting upon separation. And in Book V he suppressed the marvellous stroke of genius of Eudoxus, that his definition of equivalence also suffices to define *velocity*.

One can even hear the humourless Euclid enunciating: "Velocity is done by the astronomy department."

Historically it is difficult to find any rigorous evidence for or against the hypothesis that Eudoxus solved Zeno's problems.[6] But if we look at the problem *intrinsically* from the content of the mathematics itself as opposed to *extrinsically* from the chance observations of a philosopher like Proclus, writing seven centuries after the event, then I put it to you that the evidence becomes overwhelming. For, at the time of Eudoxus, faced with Zeno's paradoxes,

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the problem of defining velocity to the satisfaction of the philosophers must have been as formidable a task as the understanding of limits, and remember that to satisfy the philosophers was Eudoxus' particular predicament, because he was a research student in Plato's academy.

"If you can't even show us how to divide incommensurables" the philosophers would pityingly smile "how absurd to suggest that you can divide space by time."

Without the real numbers, and knowing the Greek repugnance for units—they would be most reluctant to base so fundamental a definition as velocity upon the arbitrariness of choice of unit—the problem must have seemed insoluble.

Surely this must have been the main content of Zeno's third paradox, The Arrow[3,6]:

A moving arrow at any instant is either at rest or not at rest, that is, moving. If the instant is indivisible, the arrow cannot move, for if it did the instant would immediately be divided. But time is made up of instants. As the arrow cannot move in any one instant, it cannot move in any time. Hence it always remains at rest.

The first step in resolving the paradox is to present a satisfactory rigorous definition of velocity, so that at any rate we can replace the word "moving" at the end of the first sentence by "it has a velocity." And Eudoxus can easily provide the required definition of velocity by a minor modification of his famous definition of equivalence above. For take a, b to be lengths, and a', b' to be time intervals, and define

$(a, a') \sim (b, b')$ *if the same condition holds.*

Define the *velocity* of an arrow travelling length a in time a' to be the equivalence class of (a, a'), which we again denote $a{:}a'$. There is plenty of evidence that Eudoxus was in the habit of switching terms like this, because in the special case that all four magnitudes are of the same type we find in Euclid Book V Proposition 16, the Alternando,

$$a{:}a' = b{:}b' \Leftrightarrow a{:}b = a'{:}b'.$$

Again there is plenty of evidence that he was used to situations in which two of the magnitudes were of one kind, and the other two were of another kind—witness the result from Book XII where a, b are volumes and a', b' areas. Indeed in the famous definition he explicitly omits the word ὁμογενῶν (homogeneous), which is what he would have used had they been all of the same kind.

I have said that Eudoxus could easily have provided the rigorous definition of velocity. Of course we can never know since apparently no works of his survive. All we have are the beautiful fragments in Euclid Books V, X and XII. But I put it to you, given the environment, given the problems of the time, given that famous definition, and given how easily it solves both problems of defining ratios of incommensurables *and* the definition of velocity, and given that Eudoxus' theory of proportion was known as the crown of Greek mathematics, is it not irresistible to conclude that he must have intended it to do both? One can even conjecture the very words that he would have used:

(3α) *Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν η κατὰ πηλικότητα ποιὰ σχέσις.*

(3β) *Τάχος ἐστι δύο μεγεθῶν, πρῶτον μὲν τοῦ μηκοῦς δευτέρον δὲ τοῦ χρόνου, ἡ κατὰ πηλικότητα ποιὰ σχέσις.*

The first occurs as Definition 3 of Euclid Book V (see reference 1, vol.2, p. 116), and the second is my invention* to match. My translations are

(3*a*) *Ratio is an equivalence class, with respect to size, of a pair of magnitudes of the same kind.*

(3*b*) *Velocity is an equivalence class, with respect to size, of a pair of magnitudes, the first a length and the second a time interval.*

Let us briefly summarise the first 5 definitions of Euclid Book V.

Definition 1: Multiple
Definition 2: Submultiple
Definition 3: Ratio
Definition 4: The Archimedean axiom†
Definition 5: The equivalence relation

Beautiful. Absolutely minimal. But of course I conjecture that Eudoxus had (3α) *and* (3β). And as Euclid was copying out the list for his lectures on pure mathematics his scholarly compartmentalist approach induced him to snip out (3β) and send it along with a memo to the astronomy department. And of course the head of the astronomy department, like any sensible busy experimentalist of today, flicked it straight into the waste-papyrus basket. And so it was dropped from the syllabus. And then forgotten. And sometime later the original was burnt. And so irretrievably lost. Or perhaps not quite? Moral: Never entrust the safe-keeping of research to government research establishments, but rather to many individually opinionated academics, each of whom will have his own particular prejudice of what is important. Never trust mammoth coverages of all mathematics, like Bourbaki, but carefully keep original papers, especially the collected works of great mathematicians.

Returning to Euclid, we now see a familiar phenomenon take place. Whenever you introduce a subtle new concept in a lecture course (like equivalence class), and you only introduce one example, and that the most obvious (like ratio), then your audience will understand neither the subtlety of the new concept nor why you are making such a fuss. And that is exactly what happened to (3α). It has baffled all the translators of Euclid, from Barrow to de Morgan, from Heiberg to Heath. In the latest authoritative translation into English,[1] you will find a baffled essay by Heath. And if you look at Heath's translation of (3α) it is

(3*a*′) *A ratio is a sort of relation in respect to size between two magnitudes of the same kind.*

He has interpreted it as the teacher's reassuring fatherly pat to the 5-year old learning fractions for the first time "you see a fraction is a sort of relation between numerator and denominator." But nowhere else in the whole of Euclid do you ever find a reassuring fatherly pat. In fact I even suspect that he may have been neither a father, nor a person, but a Bourbaki‡. All the other Greeks did at least have a birthplace, and began their papers with an introduction, as we do today, but Euclid

* Thanks to assistance from Hugh Dickinson over the grammar.
† Eudoxus is of course 100 years before Archimedes.
‡ Bourbaki,[10] on the other hand, if not exactly a father figure, does at least adopt the rôle of traffic cop on dangerous bends, and does at least provide an introduction. But Bourbaki clearly identifies himself with Euclid by his choice of title, and he evidently finds Euclid *très sympathiques* because in the second sentence of that Introduction to Bourbaki Book 1 Part 1 Chapter 1 we find "... what constituted a proof for Euclid is still a proof for us ..."

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begins his:

"Definition 1. A point . . ."

The difference between our two translations all hinges upon the use of the phrase "ποιά σχέσις." Heath translates it as the warm fatherly pat "a sort of relation" whereas I translate it as a cold definition "an equivalence class." To the modern mathematical ear, one translation sounds vague and the other precise, but this is merely a question of conventional jargon. If you ask 99 per cent. of people—and I tried it out on my wife—they say both phrases sound fairly vague. Similarly 99 per cent. of ancient Greeks, and probably 100 per cent. of modern Greek scholars, will say that ποιά σχέσις is vague. But Eudoxus, and the algebra school of 370 BC, may have chosen that particular phrase as their particular jargon for equivalence class. Of course what needs to be done is a careful study of Greek papers in *abstract algebra* of that period, of which probably none survives. It may not be any good examining contemporary papers in geometry, because they may not have needed to use the jargon, nor papers of 100 years later in applied mathematics, like those of Archimedes. Nor will it be much use examining the later algebra papers of the Alexandrian school, because by that time Euclid will have fouled up the whole thing as we now unfold.

The next step in our detective story concerns Zeno's 4th paradox,[3,6] the Stadium:

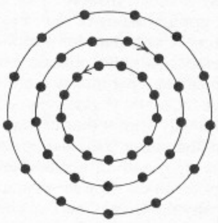


Fig. 4

Consider three rows of bodies, of which A is at rest while the other two B, C are moving with equal velocities in opposite directions. By the time they are all in the same part of the course B will have passed twice as many of the bodies in C as in A. Therefore the time it takes to pass A is twice as long as the time it takes to pass C. But the time which B and C take to reach the position of A is the same. Therefore double the time is equal to half the time.

To sort this out, all Eudoxus would have needed was the theorem

$$(a:a') : (b:b') = (a:b) : (a':b').$$

Then he would have completed the solution of all 4 of Zeno's paradoxes, the first two by ratios and the second two by velocity. But when we come to look for this theorem in Euclid Book V we find to our surprise that it is missing. We look a little closer and are shocked to find that *Euclid forgot to define the ratio of two ratios*, the kind of basic definition that anyone with an ounce of sympathy for algebra would have felt obliged to mention. Although it is needed for ratio of velocities, apparently it happened not to be needed for the geometry*, all that Euclid really cared about, and so perhaps that was why he omitted it. Then a little closer, to discover with the horror that he *couldn't* define the ratio of two ratios, because he had so messed up the order of the propositions, that he had made it impossible for himself. And at last we discover the cause of the mess—none other than the *Euclidean algorithm*, which properly lives in Book VII Proposition 2, where it is used to find the greatest common divisor of two integers. This little trick, that was obviously Euclid's only piece of research as a research student, he is eagerly awaiting to include in his lectures at the earliest possible opportunity, and he suddenly spots an opening to use a similar technique in Book V Proposition 8. But this use of the additive structure, and especially the use of *subtraction*, is unnecessary at this stage, and far too early for the theory of proportion. And putting it here wrecks the whole of Eudoxus' careful plan. Moreover it is clear that Euclid does not fully understand the theory of proportion, because he is so nervous of it—otherwise why, when he gets to his own familiar little back-garden in Book VII amongst the integers, why does he dare not use it?—and, instead, unnecessarily repeats a homely watered down version of it, that is useless for anything else.

* Mo Hirsch points out that it is needed for cross-ratios, which perhaps explains why Euclid never achieved projective geometry.

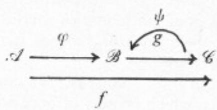
The only way to disentangle Euclid's mess is to rewrite Book V in modern category theory.[11] And then the exquisite delicacy of Eudoxus' famous definition really becomes apparent for the first time.

In effect Euclid uses Eudoxus' definition as a functor

$$\mathscr{A} \xrightarrow{f} \mathscr{C}$$

from the category $\mathscr{A}$ of sets of magnitudes to the category $\mathscr{C}$ of sets of ratios. An object $A \in \mathscr{A}$ is an ordered set with additive structure. An object $C \in \mathscr{C}$ is an ordered set with unit, inverses and Q-action, where Q denotes the positive rationals. And of course there is no feedback functor $\mathscr{C} \to \mathscr{A}$, so Euclid could not define ratios of ratios.

However now look again at Eudoxus' definition: it is really a functor defined on a much more delicate category $\mathscr{B}$, where an object $B \in \mathscr{B}$ is just an ordered set with N-action. And this category sits neatly in between $\mathscr{A}$ and $\mathscr{C}$ with forgetful functors φ, ψ feeding into it from either side. Therefore Euclid's functor can be factored, $f = g\varphi$, where g denotes the more delicate Eudoxus functor. Therefore we have the diagram



Now there is a feedback $g\psi$ enabling us to define ratios of ratios as desired.

I am not claiming that Eudoxus invented category theory, but I am convinced that he was aware of the essential mathematical content of the above diagram, otherwise why would he have chosen exactly so delicate a weapon?

If Euclid had had the same feeling for algebra that he had for geometry, then he would have given the additive structure in Book V the same royal treatment that he gave to the parallel postulate in Book I. He would have written the first half of Book V in the category $\mathscr{B}$, and the second half in the category $\mathscr{A}$. He would have postponed the crucial Proposition 8:

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$$a < b \Rightarrow a{:}c < b{:}c,$$

upon which so many of the key results at the end of the Book depend, until the second half, because it belongs in the category $\mathscr{A}$, and is actually *false* in the category $\mathscr{B}$. By proving it too early, in his eagerness to use his algorithm techniques, Euclid is forced to introduce the additive structure, and particularly subtraction, too early, and therefore, in effect, to trample the delicate category $\mathscr{B}$ out of existence.

Back to Eudoxus: of course once he had ratios of ratios he would easily have been able to reiterate the process and finish up with a most interesting algebraic object, an ordered Archimedean group, that was commutative, but *not necessarily associative*. The multiplication is defined by

$$ab = (a{:}1){:}(1{:}b).$$

If I have persuaded you that Eudoxus knew how to take ratios of ratios, then it follows inexorably that he must have possessed a form of group theory. But not the same as ours, because ours is multiplicative, whereas his was dividative*, and they are subtly different.[11] For example for us associativity is natural and commutativity exceptional, whereas with him it is the other way round. For him commutativity follows from

$$(1{:}b){:}(a{:}1) = (1{:}a){:}(b{:}1)$$

whereas there is no natural expression of associativity in terms of division. Of course it never occurs to modern group theorists to study non-associative groups, nor to look at group theory from the dividative point of view, but this is only because they absorbed multiplication and inverses with their mothers' milk, and look upon division as a secondary operation.

"But multiplication is more natural" they will insist, "because it represents composition of maps." However that is only a pure mathematician's point of view. Looking at nature, the applied mathematician is always comparing lengths, temperatures, musical notes, etc., and so, as Eudoxus would have said, perhaps ratios are more natural than products. There is a moral here, due to René Thom: just as a baby babbles in the phonemes of all the languages of the world, but after listening to its mother's replies, soon learns to babble in only the phonemes of her language, so we mathematicians, babbling in all the phonemes of mathematics should perhaps cock an ear now and then towards mother nature. More of this in Chapter 4 below.

The lost category of Eudoxus, $\mathscr{B}$, contains several interesting objects, which he may or may not have known about, such as space-time and the tangent-bundle of the reals.[11] If we apply this functor, g, to either of these, we obtain another interesting object, a non-associative extension, G, of the multiplicative group of positive reals, R_+. One can construct G from R_+ by adding on either side of each rational, r, a predecessor r_- and a successor r_+. It is as if Eudoxus is saying reassuringly to Pythagoras and Zeno: "Don't worry, Achilles can safely approach either side of a rational tortoise, but at the same time you were quite right in feeling that irrationals were less approachable."

* My research supervisor, Shaun Wylie (who is both mathematician and classical scholar) says "dividative" ought to be "divisive," but somehow I feel this conjures up the wrong overtones. Anyway he also says that bicycles ought to be called dicycles.

Conjecture: G is the unique maximal group in $g\mathscr{B}$.

The smallest non-associative group in $\mathscr{C}$ is of order 3, and is the quintessence of dividative group theory, in the same way that the smallest non-abelian group of order 6 is the quintessence of multiplicative group theory, and I am tempted to name this little group the *eudoxan*, E. The eudoxan appears as the group of ratios of any ordered set of more than one element, on which N acts trivially. Consequently E has a natural notation and multiplication table:

	$<$	$=$	$>$
$<$	$<$	$<$	$=$
$=$	$<$	$=$	$>$
$>$	$=$	$>$	$>$

There is also an additive form $\{-, 0, +\}$ of the eudoxan, because it appears as the fibre over Q of the orientation-bundle plus zero-section of the tangent bundle of any ordered set containing Q. The exponential map induces an isomorphism:

$$\{-, 0, +\} \xrightarrow{\exp} \{<, =, >\}.$$

One can argue that in the additive eudoxan *two minuses make a plus*, because the minus can be interpreted as the reversal of orientation of the underlying set. On the other hand one can argue that in the multiplicative eudoxan *two minuses make a minus*, from the multiplication table. If Eudoxus ever got an inkling of *that*, I can imagine him delightedly putting it as a paradox to all his colleagues, especially to tease the philosophers, because there is plenty of evidence that paradoxes were fashionable in those days. But alas, they probably couldn't understand, and so the only legacy that he was able to donate to his successors may have been a thorough nervousness about the minus sign, and an instinct never to touch it with a barge pole unless it was absolutely necessary. Perhaps that is why the Greeks were so chary of using the minus sign. All except Euclid, that is, who triumphantly needed it for his algorithm, and proceeded to trample with it all over Eudoxus' delicate and beautiful theory of proportion. I hope that Euclid is turning in his grave (or their graves).

I hope you will forgive my rather longwinded story of Eudoxus, which I have told for several reasons. Firstly, as you will have guessed I am a Eudoxus fan and hopeful that I might persuade one or two of you, if you have not already tried it, to blow the dust off your copies of Euclid, and follow Bolzano into the beauties of Book V. Secondly I wanted to bring to life my opening remark about mathematics being one of the oldest endeavours of man, and one through which our colleagues of yesterday can still speak to us with the freshness of today. Greek mathematics can still stimulate current research problems.

Thirdly I wanted to make a point about grand treatises like Euclid and Bourbaki, or centralised curriculum reforms. They are splendid in some places, but cannot help being biased in others. Euclid was splendid on geometry, but poor in algebra and applied mathematics. Bourbaki is splendid in algebra and analysis, but poor in geometrical thinking and applied mathematics. For example Dr. Howlett would have been as astonished by the position of Cauchy's Theorem in Bourbaki, as was Professor Penrose by that of Pythagoras' Theorem in Euclid. These treatises are trying to impose an artificial

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unity from without, that is liable to stifle growth, whereas allowing free expression to the opinions of many individual mathematicians enables the subject to evolve its own unity from within. More of this in Chapter 4. Meanwhile these grand treatises can cause mathematical loss, not only as we have seen in classical times, but even today. For compare the cautionary tales set out in Table I.

Table I

The year	The mind	The Greek	The modern
0	The discoverer	c.540 BC Pythagoras discovers that $\sqrt{2}$ is irrational.	Newton and Leibniz discover calculus, and in 1686 Newton publishes his "Principia."[12]
80	The iconoclast	c.460. Zeno poses his Paradoxes.[3-6]	In 1734 Bishop Berkeley publishes "The Analyst."[13]
160	The resolver	c.380. Eudoxus creates the theory of proportion.	Weierstrass c. 1850 and Dedekind on November 24th, 1854, create the real numbers.[1]
240	The expositor	c.300. Euclid writes his Elements.[1]	In 1939 Bourbaki begins publishing his "*Éléments*."[10]
320	The loser	By 220 the bias towards geometry has caused half the theory to be lost.	By 2020 the bias towards algebra will have caused the other half to be lost.

"How can we lose mathematics today," you may ask, and I will tell you. By the year 2010 the exponential communication explosion will probably have pushed most books off the library shelves on to tape. Soon after that, or maybe earlier, computer control will be introduced increasingly into libraries, with the automatic WC-instruction, which says: *if a tape is neither consulted, nor cited, for 10 consecutive years, then Wipe Clean.* So by 2020, sure enough, much of our mathematics will be wiped clean. And then, for the next 2000 years, our children at school will have their horizons bounded by *Bourbaki Book I, just as for the last 2000 our forefathers had theirs bounded by Euclid Book I.

Chapter 3. Teaching and research today

Turning to the present day, I should like, if I may, to put forward some tentative personal opinions. People often speak of the conflict between teaching and research, but I find the reverse. Of course there is the natural conflict with the myriad other interesting things to do in life, because the day only has 24 hours. But apart from that I find much of my teaching stimulates research, and much of my research is oriented towards teaching. For example my interest in the Greeks arose a couple of years ago, because I have to give an annual presidential address to the Polygons, our local sixth-formers association, and in order to keep the interest of the staff I search around for a new topic each year. On balance I agree with UGC policy to pay only for teaching: lecturers should be paid to lecture to the students, readers to read to them, and professors to profess to them. How well we can do this depends largely upon the staff student ratio, and ultimately upon the richness of the country. Some universities in poor countries have to cope with ratios of 1:100 or more, but, as well as teaching, the staff there still seem to manage to do a little research in the crevices of time, in the early mornings, or late at night, or on Sundays, because they deeply enjoy research—and that is the secret. We will always do research because we love it, and we will always promote people for it because they win our admiration.

* "*Théorie des ensembles.*"[10]

I find myself very much in two minds about research institutes: it is clear that the IAS at Princeton has benefited US mathematics enormously, but then the US is a rich country. In a relatively poor country like India it could be argued that the Tata Institute may have done as much harm as good, by taking many of the best mathematicians out of the universities and largely away from teaching. Of course government research establishments are a different matter, and it is clear that governments should support the development of socially useful projects like controlled fusion. But as I have tried to explain with my story of Eudoxus it would be a fundamental mistake to try and separate teaching from research, assigning teaching to the universities and research to research institutes and research establishments, as some advisers would seem to have us do in the future.

Another of my hobby-horses is the advantage of ignorance, in that it encourages creativity, both in the young and the old. May I tell a story of my first few diffident steps as a young research supervisor? I dutifully started running a seminar for my students on manifolds in about 1958, and Professor Penrose started coming along. I explained the embedding $M^n \subset R^{2n+1}$ by general position, and then conjectured that $M^n \subset R^{2n}$ because we couldn't find any counterexamples. In my ignorance I did not know that Whitney[14] had proved it 14 years previously, and that it was a well known result. The normal well educated thing would have been to equip my students with the techniques of Whitney's proof. Instead we all had a go at proving it ourselves. Roger Penrose said:

"Well if it only crosses itself in isolated double-points, why couldn't we eliminate each one, by putting a loop going off on one sheet and coming back on the other, and then putting a cone on the loop, which wouldn't meet M again if $n > 2$, and then we could slip the double-point off the top of the cone."

And I replied: "But if the loop was sort of knotted up with M, then M would get entangled with the knot as we slipped it off." And there we stuck until one day I mentioned it to Henry Whitehead over a beer. He said "That's OK by regular neighbourhoods—there's an old 1939 paper of mine that nobody ever read because* of the war." And so Penrose and I had the honour of being joint authors of Whitehead's last paper.[16] For the result of those two brief conversations was a proof, different from Whitney's, that gave, instead of one result, a whole cascade of results that reopened a chapter of geometric topology.

Another point I would like to make is that even administration can sometimes help research, not only the research of those administered to, but also that of the administrator himself—although it probably requires pretty strong self-discipline to survive as a researcher for more than 5 years of heavy duty administration. I remember when the Warwick Research Centre was being

* Actually nobody read it because it is practically unreadable.[15]

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set up, its Advisory Board thought that in addition to running symposia* in the subjects of Warwick expertise, it ought also to run one now and then in fields that were booming internationally, but by accident happened to be underrepresented in this country. Examples of such symposia were 1966/7 Harmonic analysis, 1968/9 Qualitative theory of differential equations, 1971/2 Algebraic geometry. For the hope was that we might thereby stimulate a few British mathematicians to enter those fields. And what with the administrative business of finding out what those fields actually were, and whom we ought to invite, I found myself hoist by my own petard. I became a victim of my own administrative policy.

Chapter 4. Qualitative developments in science

I was particularly intrigued by the meaning of the word "qualitative." One might define a *qualitative property* to be a diffeomorphism invariant, as opposed to a *quantitative property* which is an affine invariant. In a science in which different experimenters may use non-linearly related scales to measure the same data, only qualitative conclusions can be deduced from the resulting experimental graphs. Thus the laws of those sciences must be expressed in qualitative language. This is particularly true of the social sciences.

Now, although diffeomorphism invariants include topological invariants, many of which have been known for a long time, such as Betti numbers and homology groups, very few have in fact proved useful in describing experimental graphs. The qualitative language available for describing graphs has been of such poverty, until recently, that its existence was barely recognised. It was limited to a few words such as "increasing" or "single-valued," and as a result two things occurred. Firstly any qualitative mathematical statement about a graph could be translated so easily into everyday language, that it was not recognised as being mathematical by scientists. Secondly such statements were so obvious that they were too simple to be classed as laws, and were accused of being both trivially true, and trivially false. Let me give an example:

Hypothesis 1.

Mathematical enjoyment is an increasing function of creativity.

Or, better still, the translation into English: "the more creative the more enjoyable." Most people would admit that this is trivially true, but sometimes it can become trivially false, if one happens to be enjoying reading somebody else's work, for instance, rather than busy creating one's own.

The trouble is that the mind hops on to the statement too quickly and hops off again to consider all the exceptions. If, on the other hand, the statement has the power to arrest the mind for a while, and to synthesise a variety of phenomena, one is more ready to accept the statement as a first approximation to the truth, and even perhaps to call it a law. One is more ready to forgive the law for not being quite true in all circumstances, and rather than admit that those circumstances actually disprove the law, one is inclined to take the more lenient view that perhaps the law needs modifying occasionally. This is certainly the case with the great laws of physics such as Newton's Law of gravity or Boyle's Law for gases, both of which are false. Nevertheless we are still quite happy to call them laws, and to have them existing alongside modifications such as relativity and Van der Waals' equation. I suspect that philosophers are inclined to be a little too overimpressed by the so called laws of physics, and that social scientists a little too overawed, and I would hope that the latter should begin to approach the whole matter of laws in a more adventurous spirit. If I may be forgiven, may I quote from a recent paper with Carlos Isnard[17]:

"*A scientific law is an intellectual resting point.* It is a landing, that needs must be approached by a staircase, upon which the mind can pause, before climbing further to seek modifications."

Of course there may be more than one staircase rising from that landing. And so a science begins to grow and fork like the trunk and branches of a tree, with the forks representing the laws of that science, synthesising the ideas below them. Each science is like a grove of trees, and the most delicate features of that grove will be the blossoms and leaves, opening fresh each spring, and giving it its shape as seen from a distance. The blossoms represent the conversations between the scientists involved, and the leaves their experiments and research papers. The blossoms allow for cross-fertilisation, and the leaves provide the wherewithal to help new twigs to swell into new strong branches. And each autumn the leaves fall, providing humus to feed and strengthen the main trunks. Some trees may rot and fall, but the grove is refreshed and sustained by the appearance of new saplings, just as a science is refreshed and sustained by the appearance of new paradigms.[18] This view of science frees us from vain attempts to impose artificial unity from without, and allows us to admit to an open ended concept of unity evolving from within.

"It's all very well waxing lyrical" the social scientist will reply "but the poverty of the qualitative language offered to us by the mathematicians makes it very difficult to get any kind of tree off the ground, let alone a grove." But that is just where the social scientists might be wrong, because there is a new fertiliser on the market, called catastrophe theory,[17,19] ideally suited for stimulating the growth of new paradigms.

Catastrophe theory substantially enriches the language of diffeomorphism-invariants with words that are more subtle, neither trivial, nor so easily translatable into non-mathematical language. And they have impressive power of synthesis. In fact if I had to select one extra tool out of the whole of mathematics to add to the Hammersley tool-kit, I would choose the cusp catastrophe, pictured in Fig. 5. The cusp catastrophe describes the canonical way that two control factors can interfere with one another when influencing the same behaviour mode.

Chapter 5. Why mathematics is sometimes exciting and sometimes dreary

To conclude the paper let me suggest one application of the cusp catastrophe that synthesises much of the intuition about the teaching of research students described by previous speakers. This example is an extension of Hypothesis 1 above, but instead of being trivial it fulfills the requirement of giving the mind pause. It is a light hearted example, with no claim to be a law, and mathematically it is not as serious as the social science

* Symposia are named after Plato s original idea of the Symposium, which was great conversation over drinks.

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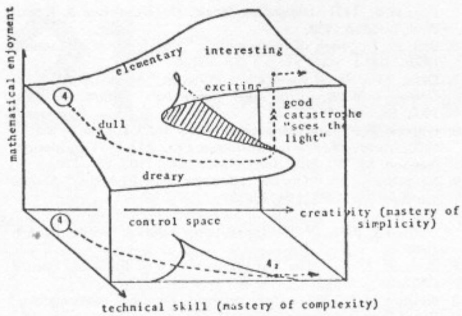

Fig. 5. Graph

models in reference 17, because there is no dynamic maximising the probability, nor suggested design of experiment*. Therefore the surface pictured above represents only the most probable behaviour rather than the actual behaviour. The purpose of the example is to summarise, to give insight, and hopefully to contain a few germs of truth.

Hypothesis 2.

Creativity (mastering of simplicity) is a normal factor, and technical skill (mastering of complexity) is a splitting factor, influencing mathematical enjoyment.

The definition of normal and splitting factors are given in reference 17, but for our purposes it is sufficient to say that the hypothesis means that the graph of enjoyment, as a function of creativity and technique, looks like Fig. 5. In particular in elementary mathematics, where little technique is needed, Hypothesis 1 becomes a special case of Hypothesis 2, and so the latter is a generalisation of the former. Meanwhile the acquisition of technique has a splitting effect, causing the function to become double-valued, and the graph to become double-sheeted into exciting and dreary mathematics. The middle shaded piece of the graph represents least probable situations, and so is irrelevant—we only use the upper and lower sheets.

We now view teaching techniques at various levels by the paths shown in the control space Fig. 6. In each case the resulting enjoyment of the students can be traced by lifting the path to the surface in Fig. 5. For example the path (4) is shown lifted, with a "good" catastrophe occurring above the point 4_2.

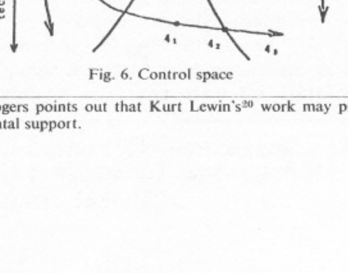

Fig. 6. Control space

* Leo Rogers points out that Kurt Lewin's[20] work may provide experimental support.

(a) School children learning tables*

Path (1) represents the old-fashioned drilling technique, which became very dreary. Path (2) represents the opposite approach of allowing children only to play, which was quite enjoyable, but did not give much skill. Path (3) represents the better modern approach, which allows children first to discover multiplication for themselves, by playing with stones in egg containers for instance, and then encouraging them to fill excitedly exercise books, mastering the technique. At least in early primary education (and before the prison doors of Bourbaki's set theory have had time to clang shut in secondary education) we seem at least to be returning a little towards the spirit of the pre-Euclidean Greeks, who believed that education should be enjoyable. For instance Plato writing in the Laws,[3] says:

"First there should be calculations specially devised as suitable for boys, which they should learn with amusement and pleasure, for example, distributions of apples or garlands where the same number is divided among more or fewer boys, . . ."

The three basic ingredients of school mathematics should be geometric intuition, physical intuition, and a sense of fun. Then the fourth ingredient, a sense of rigour, will grow of its own accord.

(b) Undergraduate lecture courses

Path (1) represents a bad lecturer giving bad material, which the students find first dull and then dreary. Path (3) represents a good lecturer giving good material, which the students find first interesting and then exciting. Path (4) represents a bad lecturer with good material—the sort of lecturer who spends most of the course setting up the machinery without giving any motivation, and then brings it all together to prove the major theorems in the last few lectures. The good students suddenly see the light at point 4_2 and jump catastrophically from the lower sheet to the upper sheet, from the dreariness of the machinery, 4_1, to the excitement of the theorem, 4_3. But, alas, the bottom half of the class, who generally fall behind towards the end of a lecture course, get stuck at the dreary point 4_1, and never see the light.

Finally path (5) represents the good lecturer giving bad material. The students enjoy the lectures at the time, 5_1, but when they come to revise it, at 5_2, they suddenly realise the material is pretty dull, and jump catastrophically from the upper sheet to the lower sheet, finishing somewhat disillusioned at 5_3.

(c) Research students and research supervisors

Path (1) represents a poor student with a poor supervisor, writing a dreary thesis. Path (3) represents a good student with a good supervisor writing an exciting thesis. Path (4) represents a good student with a poor supervisor, who writes a dull thesis at 4_1, made of tin, but once he gets free of his supervisor, at 4_2, suddenly blossoms into doing exciting research at 4_3. Path (5) represents a poor student with a good supervisor, who imparts a spurious creativity to the student during his PhD years, causing him to write an interesting thesis at 5_1, made of gold, but once he gets free of his supervisor, at 5_2, he collapses into writing dull papers using the same old techniques.

(d) Mathematicians in a rut

A mathematician may have bravely started on path (3),

* I am indebted to Ruth Rees for this example.

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and written several good papers, but the path may then bend round into (5) as he reaches the limit of his creativity in that field, or gets bored with the subject, or gets imprisoned in the very techniques that he himself has created. Once he appreciates that he has suffered the bad catastrophe at 5_2, and is now in a rut at 5_3, there is only one thing to do: change fields. This is what Professor Bondi was recommending even as early as immediately after the PhD. Since our mathematician has no technique in the new field, he jumps straight back to the beginning of, we hope, path (3). It is very important that he soon tries his hand at a little creativity in the new field, in order to round the right hand side of the cusp, before learning all the new techniques. Otherwise those techniques will prove not interesting, but increasingly dreary as he follows path (4), perhaps to get stuck at 4_1. The more technical skill he acquires in the new field, the further away recedes the point 4_2, because the cusp is ever widening, and so the more creative he is required to become before he can ever have a chance of doing anything exciting. This was the serious intent behind my apparently flippant remark above, in favour of ignorance, and why I told the story of Whitehead's last paper.

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