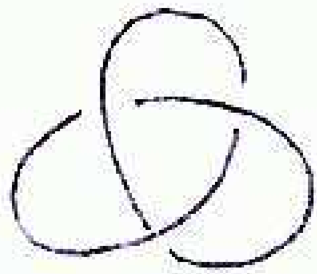
Definition

A <u>knot</u> is a closed curve in 3 dimensions.

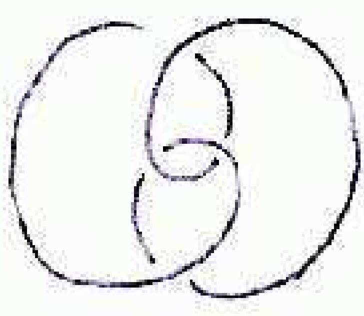


Trefoil

A <u>picture</u> of a knot consists of a finite number of arcs and crossings.

Definition

Two knots are <u>equal</u> if one can be moved into the other.



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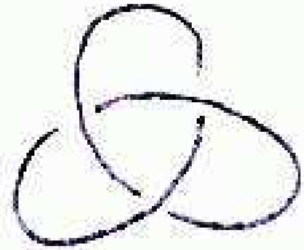
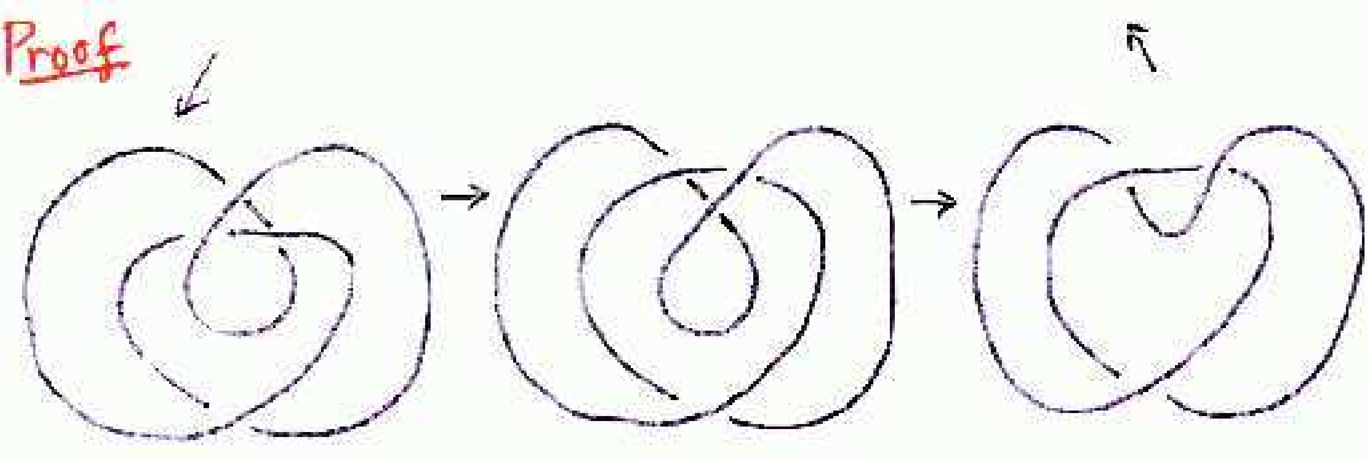


Trefoil

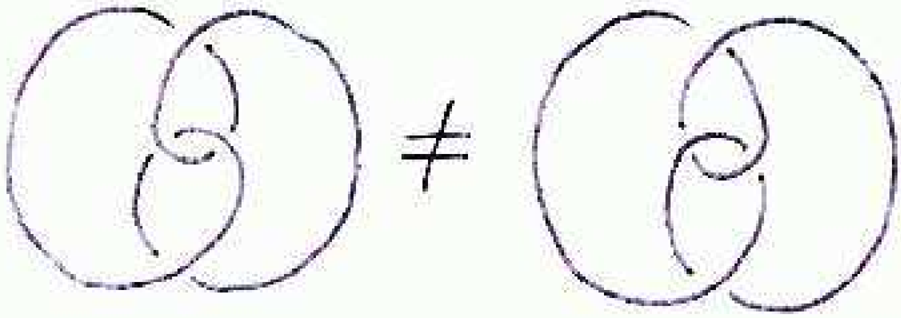
Proof



Definition

Two knots are <u>unequal</u> if one cannot be moved into the other

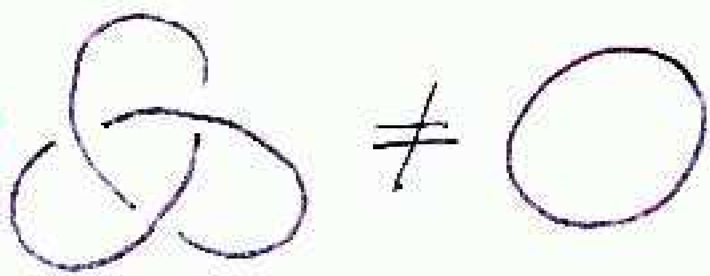
Trefoil Square-knot



Question

How do you prove this?

Trefoil



Circle

Question

How do you even prove a trefoil is knotted?

Answer

To prove two knots are <u>equal</u> use geometry.
To prove two knots are <u>unequal</u> use algebra.
We have to invent some <u>invariant</u>,
in other words a property that
does not vary if the knot is moved.

Example of an invariant

3 – COLOURABILITY

Choose 3 colours

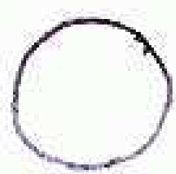
Let K be a picture of a knot (with a finite number of arcs and crossings).

Definition.
We say K can be 3-coloured if the arcs can be coloured so that
Axiom ① We use at least 2 colours
Axiom ② At each crossing we use 1 or 3 colours.

Lemma 1 The trefoil can be 3-coloured

Proof 

Lemma 2 The circle can't be 3-coloured

Proof 

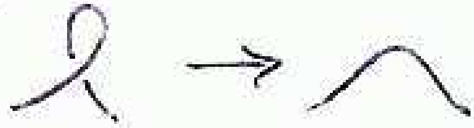
Theorem 1
If $\left\{ \begin{array}{l} \text{K can be 3-coloured} \\ \text{K is moved into L} \end{array} \right\}$ then L can be 3-coloured.

Theorem 2 The trefoil is knotted.

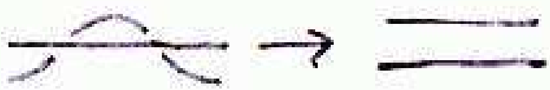
Proof of Theorem 1

Consider elementary moves of the following type:

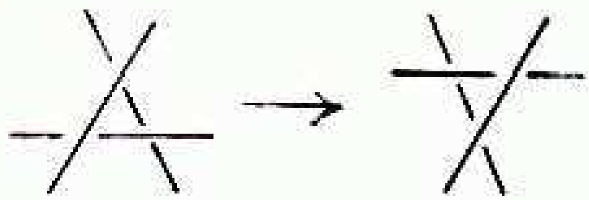
Type I



Type II

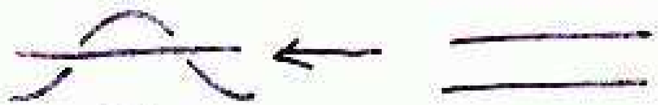


Type III

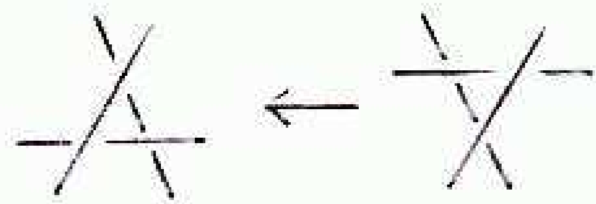


Type I inverse

Type II inverse



Type III inverse



Suppose we are given a long complicated move $K \to L$.

Imagine taking a film of it, and examining the changes frame by frame.

Each frame will represent either no change from the previous frame or an elementary move.

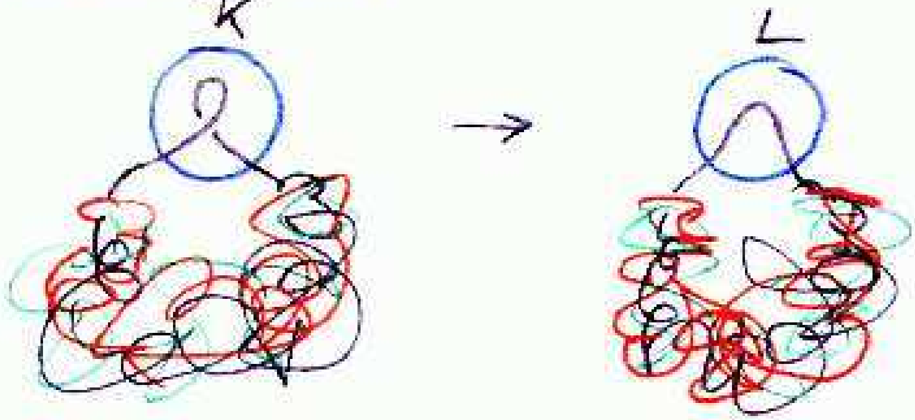
Hence we get a sequence of elementary moves

$$K = K_1 \to K_2 \to K_3 \to \cdots \to K_n = L.$$

Therefore it suffices to prove:

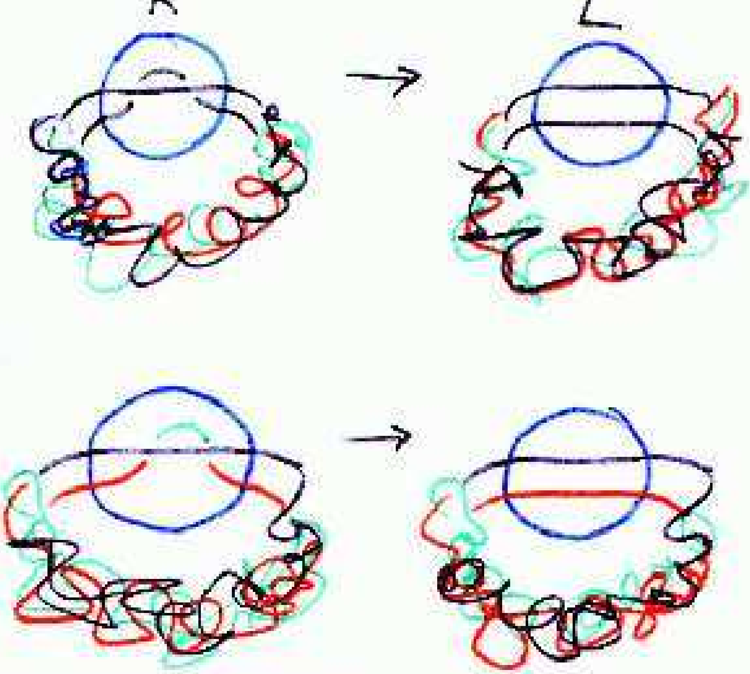
if $\left\{\begin{array}{l} K \text{ can be 3-coloured} \\ K \to L \text{ is an elementary move} \end{array}\right\}$ then L can be 3-coloured.

Type I elementary move.



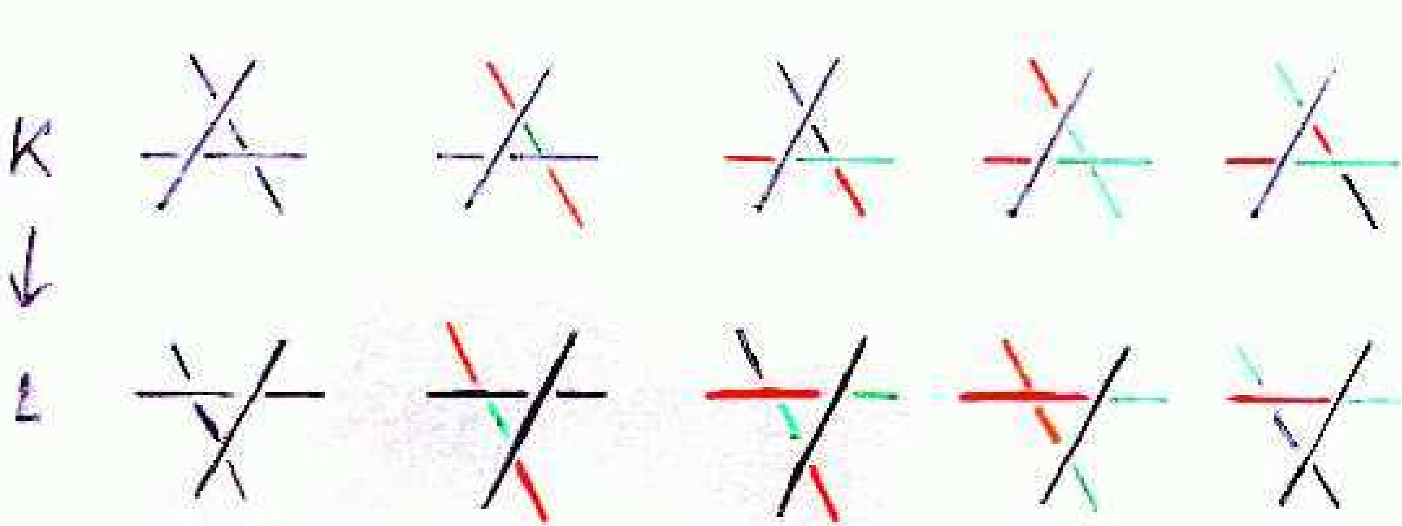
leave the colouring of the rest of the knot unchanged.

Type II. There are two possibilities.



The inverses are similar

Type III There are five possibilities



Theorem 3

You can't 3-colour the square-knot. 

Proof. Suppose you try.

There are 4 arcs and only 3 colours.

Therefore 2 arcs must be the same colour 

But any 2 arcs meet at some crossing.

Therefore the third arc at this crossing must be the same colour by Axiom ②.

Similarly the fourth arc must be the same colour.

Therefore all 4 arcs are the same colour.

This violates Axiom ①.

Therefore you can't 3-colour the square-knot.

Remark You can 3-colour the four-crossing picture of the trefoil. 

Corollary Square knot $\neq$ trefoil

Problem How do you prove the square-knot is knotted?

Answer We generalise the invariant.

p-COLOURING

Let p be an odd prime. For example 3, 5, 7, 11, 13, ...

We shall use as "colours" the integers mod p:

$$0, 1, 2, \ldots, p-1.$$

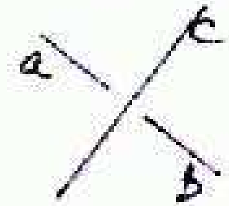
Let K be a picture of a knot.

Definition. We say K can be <u>p-coloured</u> if

Axiom ①. We use at least 2 colours

Axiom ② At each crossing the overpass is the average of the two underpasses (mod p)

$$a+b \equiv 2c \pmod{p}.$$



Here "equals mod p" means the two numbers are the same or differ by a multiple of p.

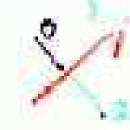
We call p a <u>code</u> of K.

<u>Example $p=3$</u> This is the same as we had before.

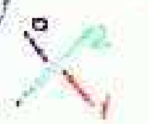
The three "colours" are 0, 1, 2

0 is the average of 1 and 2 because $1+2 \equiv 2\times 0 \pmod{3}$

1 is the average of 0 and 2 because $0+2 = 2\times 1 \pmod{3}$



2 is the average of 0 and 1 because $0+1 = 2\times 2 \pmod{3}$



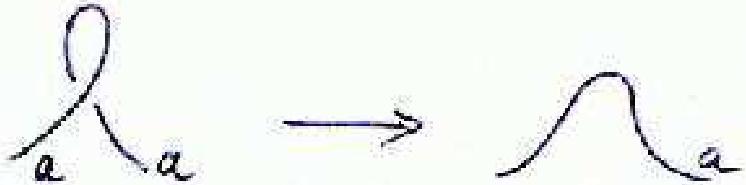
Therefore there are 1 or 3 colours at a crossing.

Theorem 4. Codes are invariant

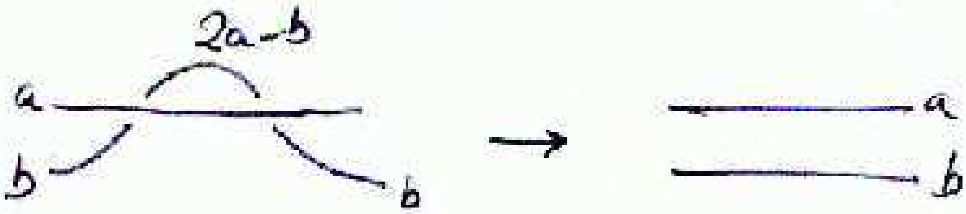
Proof. It suffices to show that

if $\left\{\begin{array}{l} K \text{ can be } p\text{-coloured} \\ K \to L \text{ is an elementary move} \end{array}\right\}$ then L can also be.

Type I

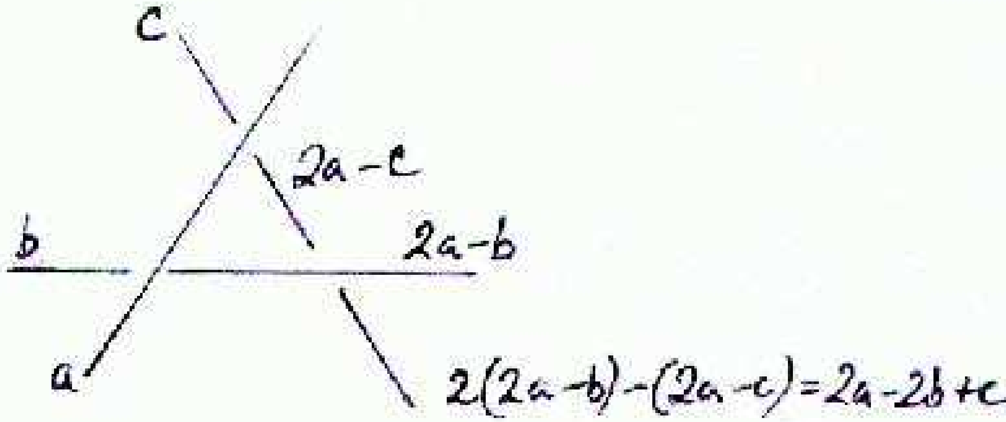


Type II

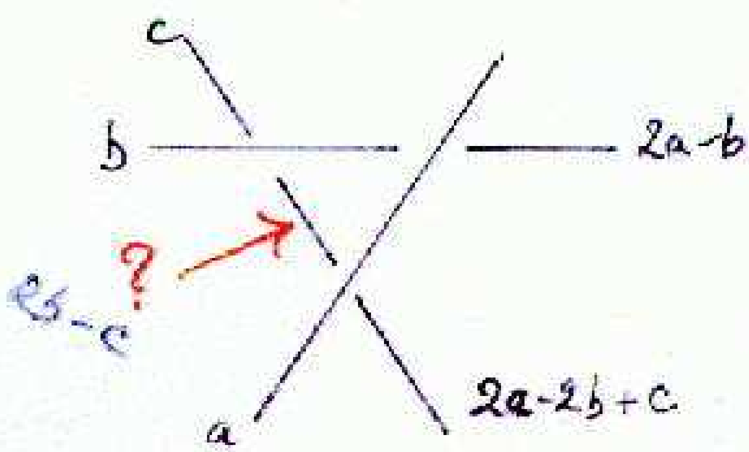


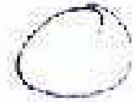
Type III

K, Before

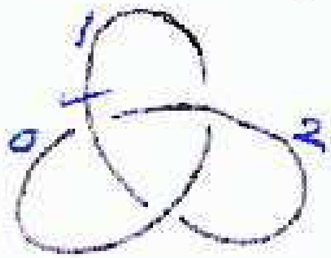


L, After



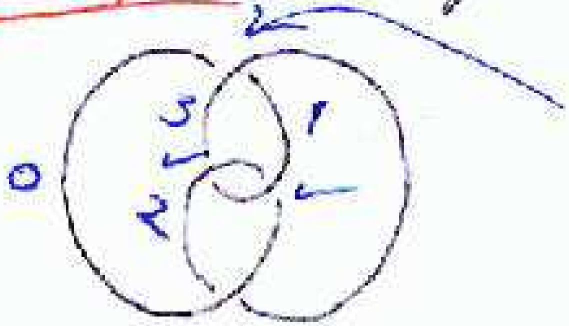
Example 1 The circle has no codes. 

Example 2 The trefoil has code 3.



$0+1 = 2\times 2 \pmod p$
$0 = 3 \pmod p$
$p=3$

Example 3 The square-knot has code 5.



$0+1 = 2\times 3 \pmod p$
$0 = 5 \pmod p$
$p=5$

Therefore the square-knot is knotted & is not equal to the trefoil

Question Have they any other codes?

Answer No

(because if you can p-colour then you can always recolour so that two particular arcs have colours 0, 1).

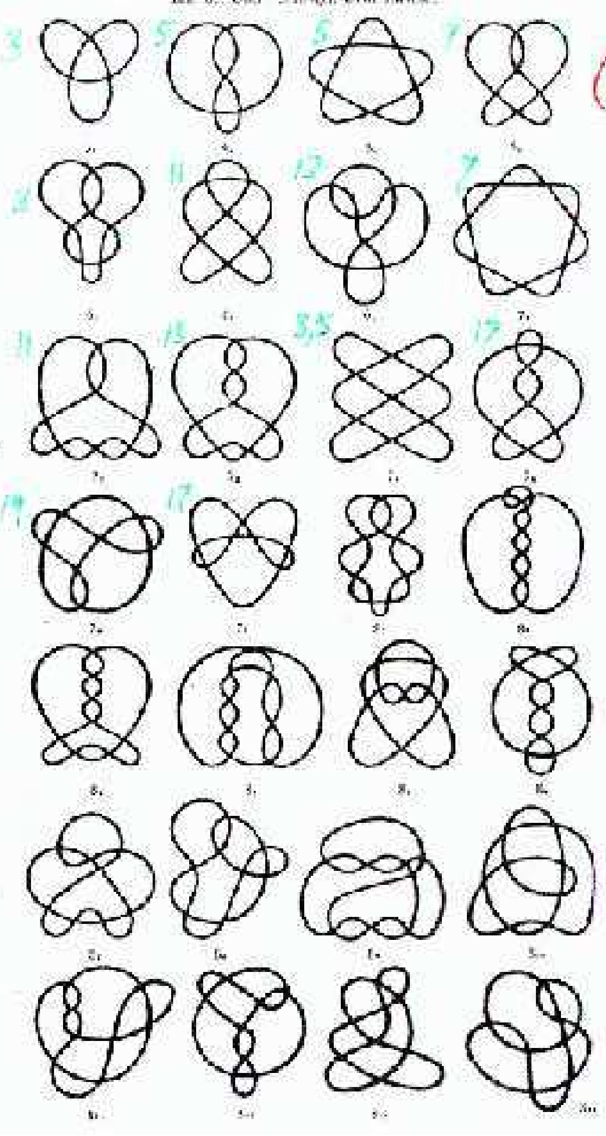
Question Can a knot have more than 1 code?

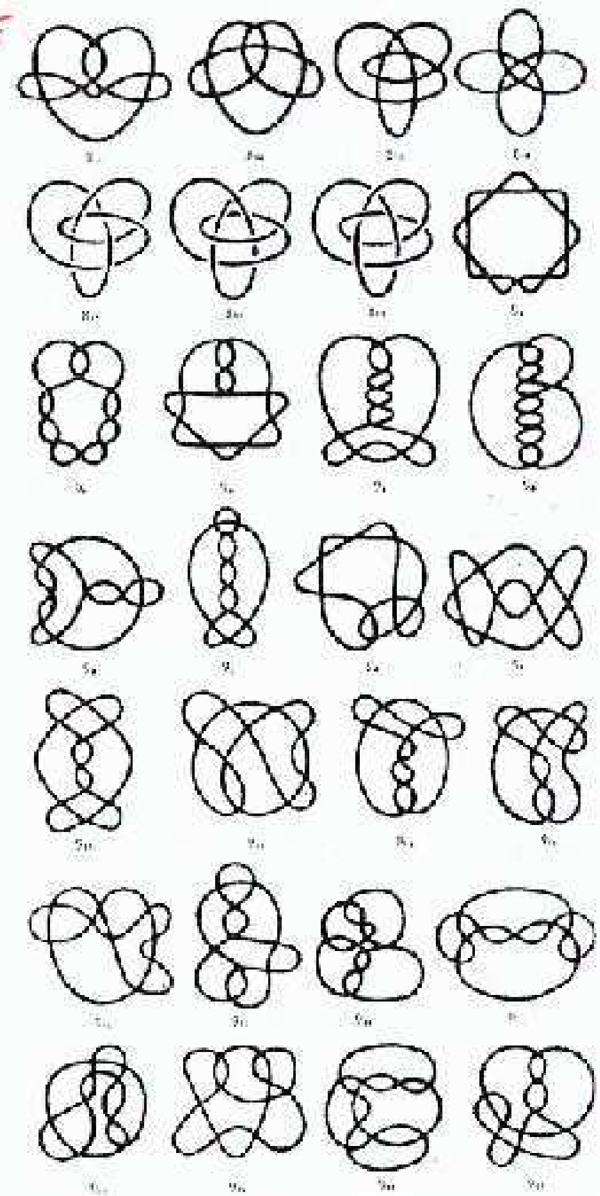
Answer Yes. The product of the trefoil & the square-knot has codes 3 & 5

REIDEMEISTER
KNOTENTHEORIE
1948
(Chelsea)

Knotentabelle.

[illegible]





Theorem 5 Any knot has only a finite number of codes.

Proof

Let K be a picture of a knot.

Going round K label the

arcs $a_1, a_2, \dots, a_n$

crossings $c_1, c_2, \dots, c_n$

so that c_i is the front end of a_i.

Define an $n \times n$ matrix M with

row = crossing c_i

columns = arcs a_j

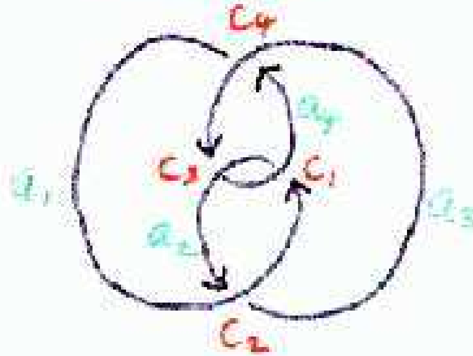
$$M_{ij} = \begin{cases} 1 & \text{if } c_i \text{ has underpass } a_j \\ -2 & \text{if } c_i \text{ has overpass } a_j \\ 0 & \text{otherwise} \end{cases}$$

Define D by crossing out the last row & column

Let d = determinant D

Then Theorem 5 follows from:

EXAMPLE.



$$M = \begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline c_1 & 1 & 1 & 0 & -2 \\ c_2 & -2 & 1 & 1 & 0 \\ c_3 & 0 & -2 & 1 & 1 \\ c_4 & 1 & 0 & -2 & 1 \end{array}$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$d = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 3 - (-2)$$

$$= 5$$

Theorem 6

The codes of K are the prime factors of d.

Proof of Theorem 6

d is odd, because $d \equiv 1 \pmod 2$

Suppose p is a code of K.

Choose a p-colouring of K.

Let x_i be the colour of a_i

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Then $Mx \equiv 0 \pmod p$

We can recolour so that $x_n = 0$ (by subtracting x_n from each colour)

Let $y = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$

Then $Dy \equiv 0 \pmod p$

But $y \not\equiv 0 \pmod p$ by Axiom 1

Therefore $d \equiv 0 \pmod p$

$\therefore$ d is a multiple of p.

$\therefore$ p is a prime factor of d.

$\therefore$ codes of K are prime factors of d.

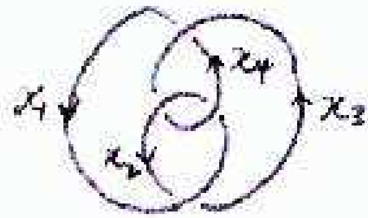
(and conversely)

EXAMPLE

$$D = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\therefore D \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \pmod 2$$

$\therefore$ $d \equiv 1 \pmod 2$



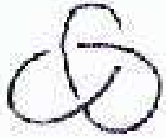
$$Mx = \begin{pmatrix} 1 & 1 & 0 & -2 \\ -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

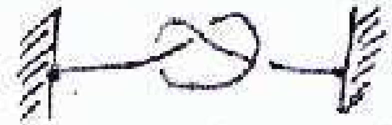
$$= \begin{pmatrix} x_1 + x_2 - 2x_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Dy = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

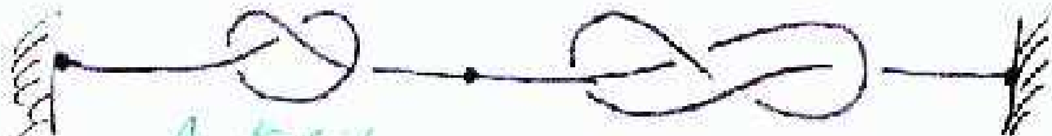
Theorem 7 Knots form a semi-group.

Proof

To each knot  there corresponds a knotted curve

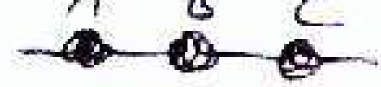


Define the <u>product</u> $A \times B$ of two knotted curves by juxtaposition



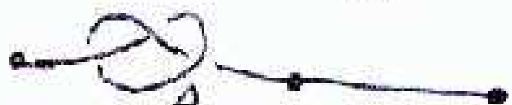
Verify that the product is

① Associative $(A \times B) \times C = A \times (B \times C)$



② Commutative $A \times B = B \times A$

③ Has a unit $A \times 1 = A$



Question Do they form a group?

In other words, given A is there an inverse A^{-1} such that $A \times A^{-1} = 1$?

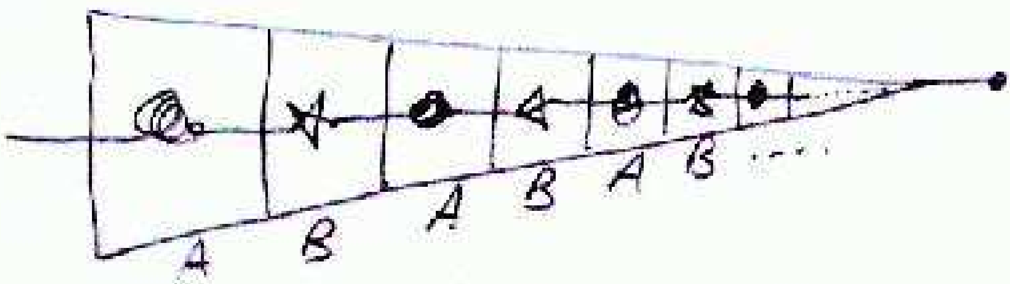
Answer No

Theorem 8 If $A \times B = 1$ then $A = B = 1$.

Proof of Theorem 8

Suppose $A \times B = 1$

Let $X = A \times B \times A \times B \times A \times B \times \cdots$



Then $X = (A \times B) \times (A \times B) \times (A \times B) \times \cdots$

$= 1 \times 1 \times 1 \times \cdots$

$= 1$

Also $X = A \times (B \times A) \times (B \times A) \times \cdots$

$= A \times (A \times B) \times (A \times B) \times \cdots$

$= A \times 1 \times 1 \times 1 \times \cdots$

$= A$

Therefore $A = 1$

Similarly $B = 1$.

Definition.

Call a knot P **prime** if $P = A \times B$ implies either $A = 1$ or $B = 1$.

Examples

The trefoil and square-knots are prime.

Theorem 9

Any knot K can be uniquely factored into primes:

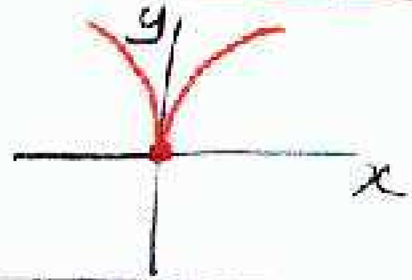
$$K = P_1 \times P_2 \times \cdots \times P_n.$$

Therefore knots are like positive integers because they have

$$\boxed{\begin{array}{l}\text{products}\\ \text{associativity}\\ \text{commutativity}\\ \text{unit}\\ \text{primes}\\ \text{unique factorization}\end{array}}$$

APPLICATION: SOLUTIONS OF EQUATIONS

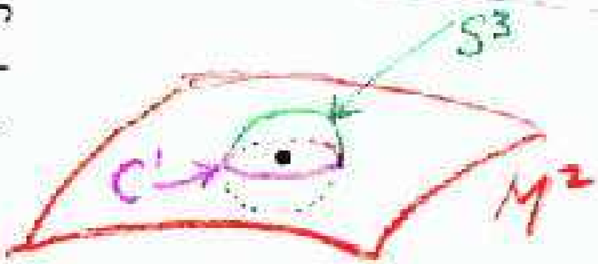
Example ① $x^2 = y^3$; $x, y \in \mathbb{R}$



Cusp (differential invariant)

Example ② $x^2 = y^3$; $x, y \in \mathbb{C}$

Solution = surface $M^2 \subset \mathbb{C}^2 = \mathbb{R}^4$



Question What does M look like at the origin?

Answer Take a little ball around the origin, D^4.
Take its surface $S^3 = \partial D^4$
Then M^2 meets S^3 in a curve C^1

Question What does C^1 look like in S^3?

Answer Its a trefoil knot.

$\therefore$ M^2 is locally knotted at the origin.

Conclusion A complex cusp is a <u>topological</u> invariant (much deeper than a real cusp).

Example ③ $x^2 + y^3 + z^5 = 0$; $x, y, z \in \mathbb{C}$.

Solution = surface $M^4 \subset \mathbb{C}^3 = \mathbb{R}^6$.

At the origin M is locally knotted with a knot of S^3 in S^5.

HIGHER DIMENSIONS

Notation

Define the <u>$(n+1)$-disk</u> D^{n+1} in $\mathbb{R}^{n+1}$ by

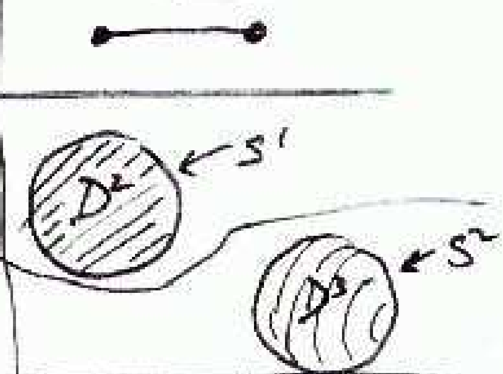
$$x_1^2 + \cdots + x_{n+1}^2 \leq 1.$$

Define the <u>n-sphere</u> $S^n = \partial D^{n+1}$, the boundary of D^{n+1},

given by $x_1^2 + \cdots + x_{n+1}^2 = 1$

Examples

n	disk D^{n+1}	sphere S^n
0	D^1 = interval	S^0 = pair of points
1	D^2 = disk	S^1 = circle
2	D^3 = ball (solid)	S^2 = sphere (hollow surface)



Remark

Circles link & knot in $\mathbb{R}^3$

Question

In what dimension do spheres link & knot?

n	LINKING is a $2n+1$ phenomenon	KNOTTING is a codimension 2 phenomenon	UNKNOTTING is a codimension 3 phenomenon
1	Two circles link in $\mathbb{R}^3$ because $2(1)+1=3$	A circle knots in $\mathbb{R}^3$ because $1+2=3$	A circle unknots in $\mathbb{R}^4$ because $1+3=4$
2	Two spheres link in $\mathbb{R}^5$ because $2(2)+1=5$ Classical	A sphere knots in $\mathbb{R}^4$ because $2+2=4$ Emil Artin 1924	A sphere unknots in $\mathbb{R}^5$ because $2+3=5$ Zeeman 1961
50	Two S^{50}'s link in $\mathbb{R}^{101}$	An S^{50} knots in $\mathbb{R}^{52}$	An S^{50} unknots in $\mathbb{R}^{53}$

LINKING

$\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$

Take a circle $S^1 \subset \mathbb{R}^2$

Let D^2 be transversal to S^1

Let $S^1_* = \partial D^2$

Then S^1_* links S^1 in $\mathbb{R}^3$

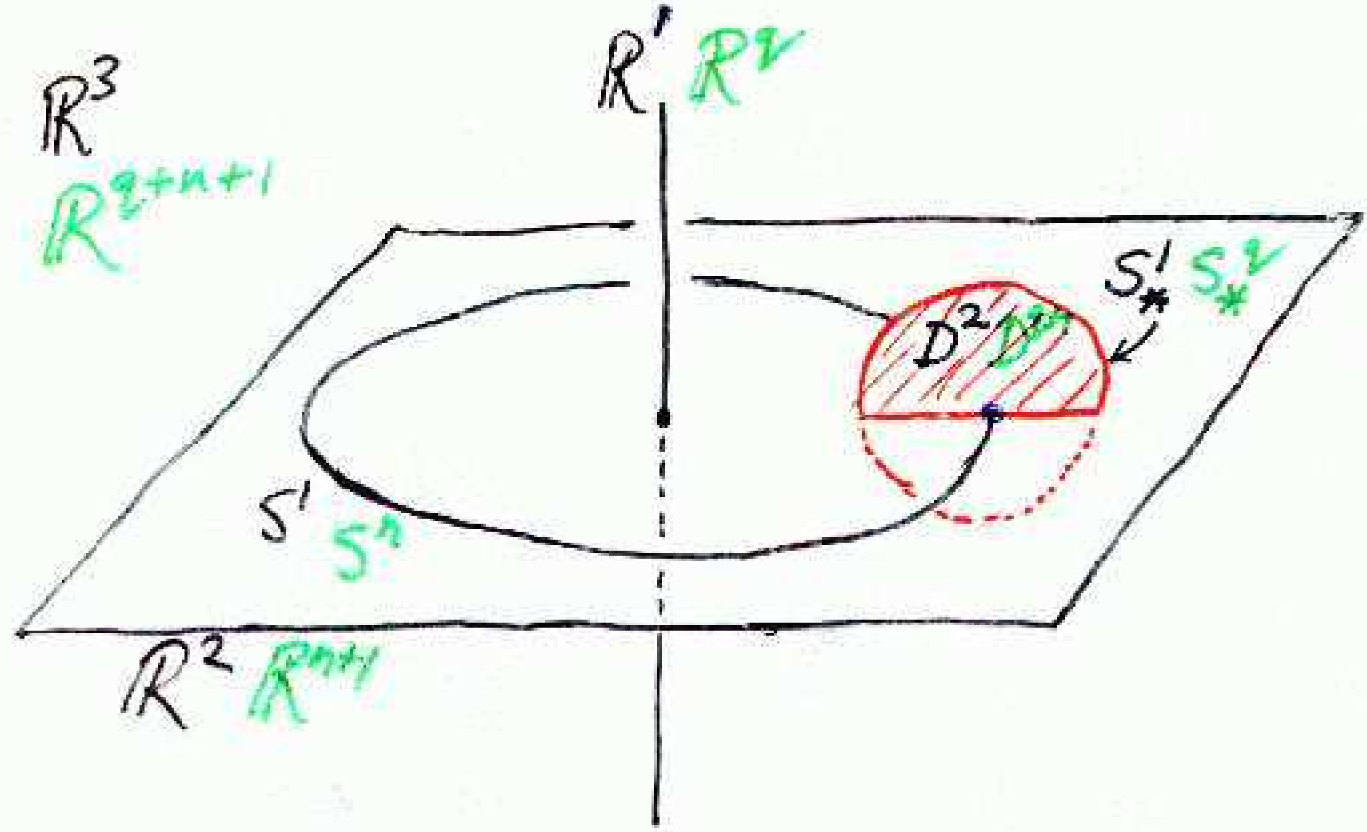
$\mathbb{R}^{q+n+1} = \mathbb{R}^q \times \mathbb{R}^{n+1}$

Take a sphere $S^n \subset \mathbb{R}^{n+1}$

Let D^{q+1} be transversal to S^n

Let $S^q_* = \partial D^{q+1}$

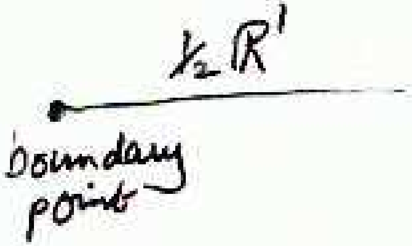
Then S^q_* links S^n in $\mathbb{R}^{q+n+1}$



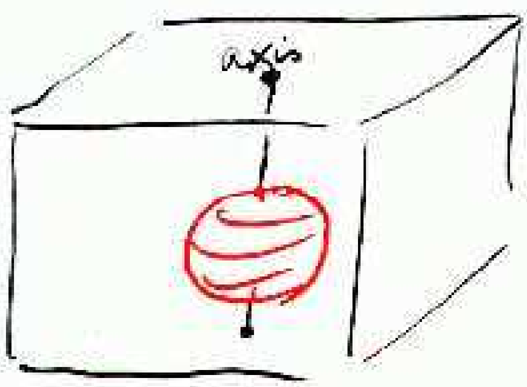
Example

S^2_* links S^2 in $\mathbb{R}^5$

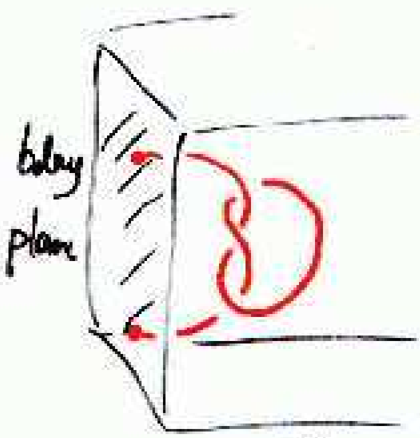
KNOTTED SPHERE IN $\mathbb{R}^4$



$$\text{spin } \tfrac{1}{2}\mathbb{R}^1 = \mathbb{R}^2$$



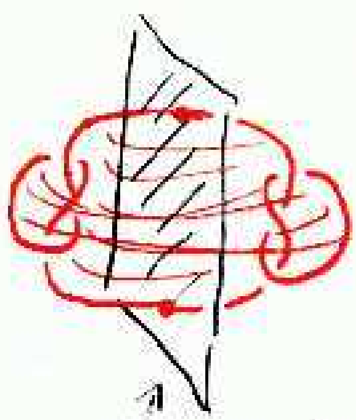
$$\begin{array}{ccc} \text{spin } \tfrac{1}{2}\mathbb{R}^2 & = & \mathbb{R}^3 \\ \cup & & \cup \\ \text{spin } A & = & S^2 \end{array}$$



$\frac{1}{2}\mathbb{R}^3$

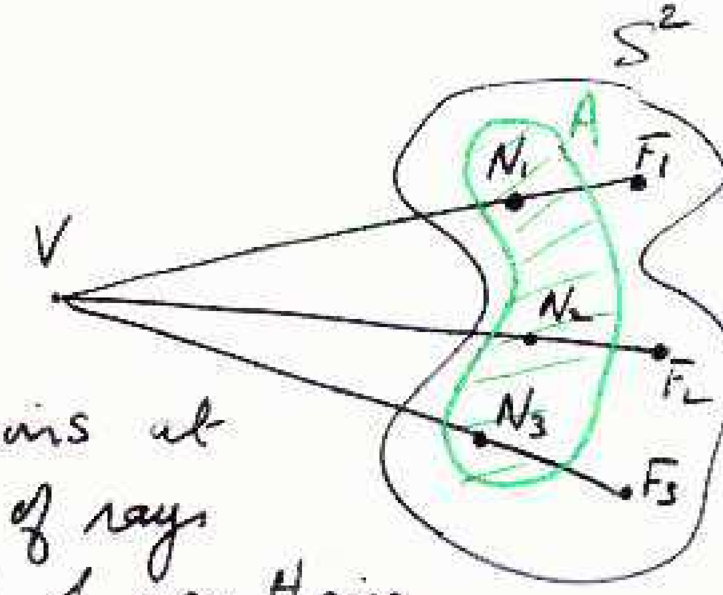
$\cup$

knotted arc K



$$\begin{array}{ccc} \text{spin } \tfrac{1}{2}\mathbb{R}^3 & = & \mathbb{R}^4 \\ \cup & & \cup \\ \text{spin } K & = & \text{knotted } S^2 \end{array}$$

THEOREM Any S^2 in $\mathbb{R}^5$ is unknotted.



Proof Given $S^2 \subset \mathbb{R}^5$.

Let V be a vertex in general position.

From dimensional considerations at most a finite number of rays through V meet S^2 twice, & none thrice.

$\therefore$ each such ray meets S^2 in a near point N & a far point F.

Let A be a disk in S^2 enclosing all near points and no far points.

Let $B = \text{closure}(S^2 - A)$

The cone VA is non-singular, because it contains no far points.

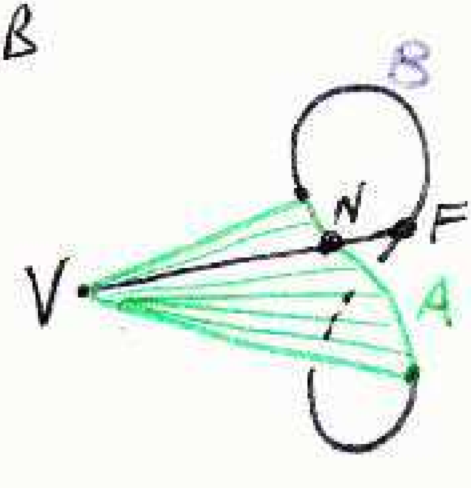
$\therefore S^2 = A \cup B$

$\sim V(\partial A) \cup B$, sliding A across VA to $V(\partial A)$

$= V(\partial B) \cup B$, since $\partial A = \partial B$

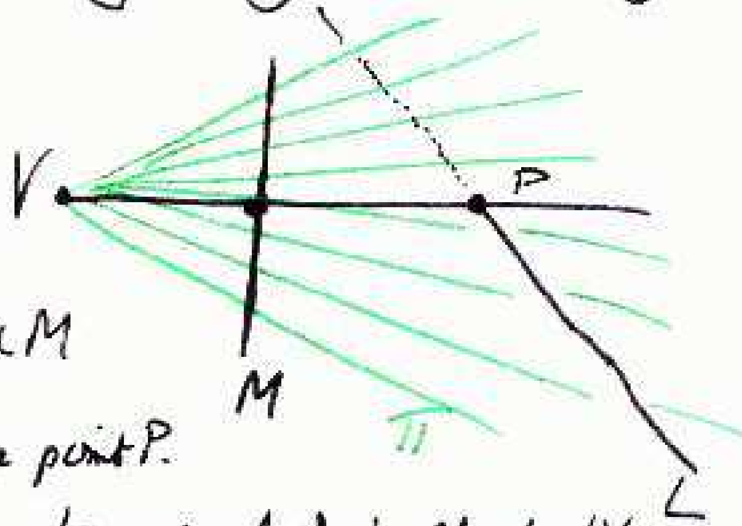
$= \partial(VB)$

$=$ unknotted because it bounds the disk VB.



DIMENSIONAL CONSIDERATIONS

Lemma In $\mathbb{R}^3$, given two lines and a vertex V in general position, then $\exists$ unique ray through V meeting both lines.



Proof

Let L, M be the lines

Let Π = plane containing V & M

Then L pierces Π at a unique point P.

Therefore LP is the unique transversal of L, M through V.

Corollary 1 $\nexists$ ray through V meeting 3 given lines.

Corollary 2 A projection of a knotted curve in $\mathbb{R}^3$ has generically a finite number of crossings & no triple points.

Lemma In $\mathbb{R}^5$, given two planes and a vertex V in general position, then $\exists$ unique ray through V meeting both planes.

Proof as above.

Corollary 1 $\nexists$ ray through V meeting 3 given planes.

Corollary 2 The projection of a knotted sphere in $\mathbb{R}^5$ from V has only a finite number of crossings & no triple points.
