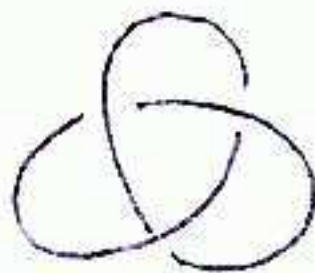


Definition

A knot is a closed curve in 3 dimensions.

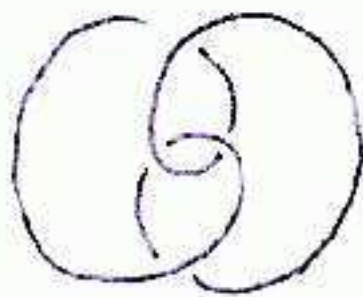


Trefoil

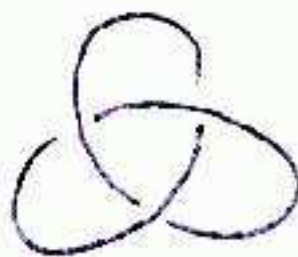
A picture of a knot consists of a finite number of arcs and crossings.

Definition

Two knots are equal if one can be moved into the other.

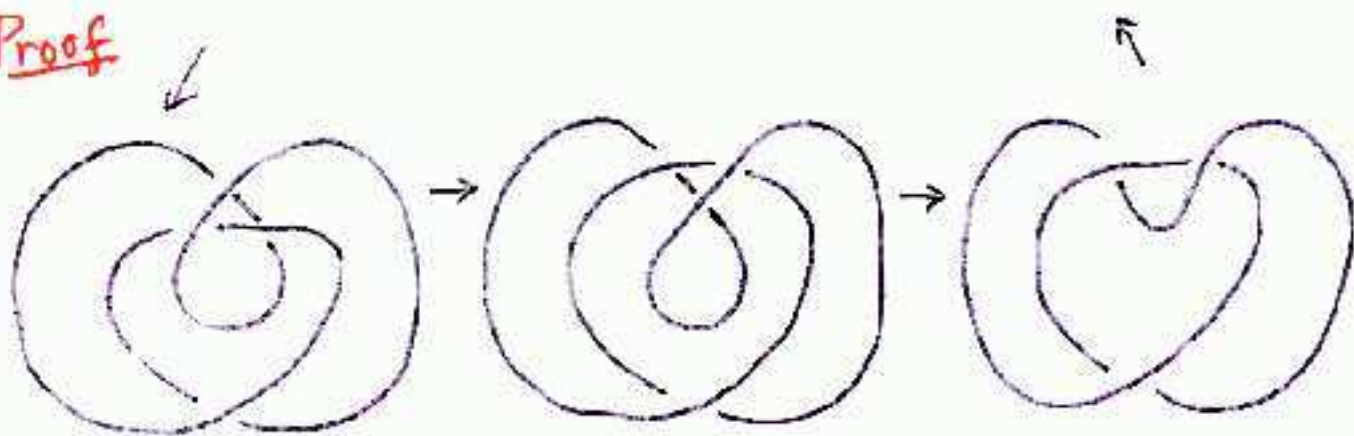


=



Trefoil

Proof

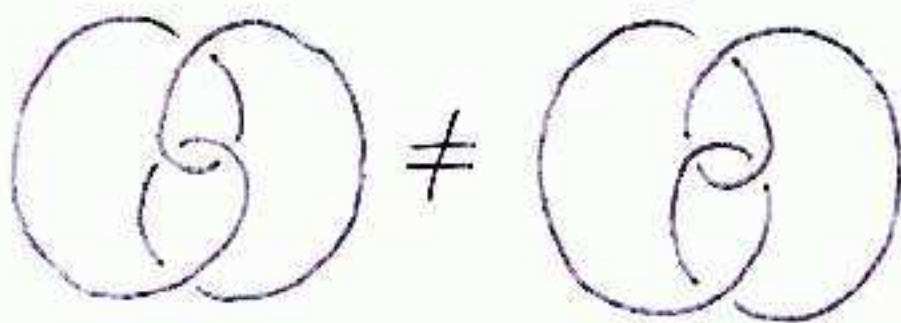


Definition

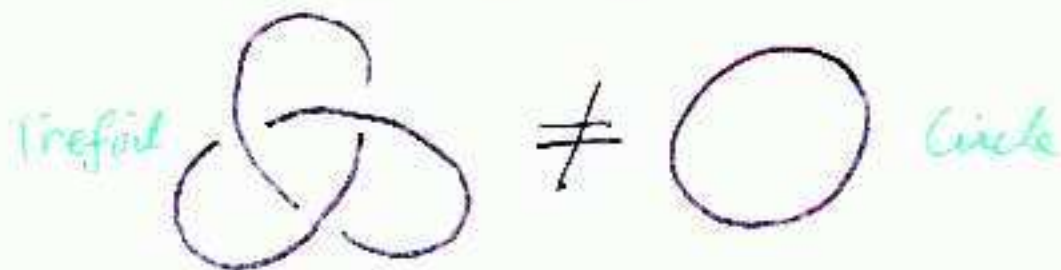
Two knots are unequal if one cannot be moved into the other.

Trefoil

Square-knot



Question How do you prove this?




Question How do you even prove a trefoil is knotted?

Answer To prove two knots are equal use geometry.
To prove two knots are unequal use algebra.
We have to invent some invariant,
in other words a property that
does not vary if the knot is moved.

Example of an invariant

3-COLOURABILITY

Choose 3 colours 

Let K be a picture of a knot (with a finite number of arcs and crossings).

Definition.

We say K can be 3-coloured if the arcs can be coloured so that


Axiom (1) We use at least 2 colours

Axiom (2) At each crossing we use 1 or 3 colours.

Lemma 1 The trefoil can be 3-coloured

Proof 

Lemma 2 The circle can't be 3-coloured

Proof 

Theorem 1

if $\left\{ \begin{array}{l} K \text{ can be 3-coloured} \\ K \text{ is moved into } L \end{array} \right\}$ then L can be 3-coloured.

Theorem 2

The trefoil is knotted.

Proof of Theorem 1

Consider elementary moves of the following type:

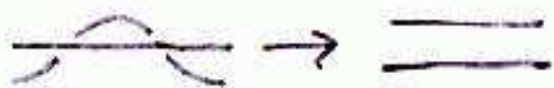
Type I



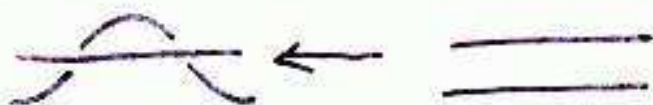
Type I inverse



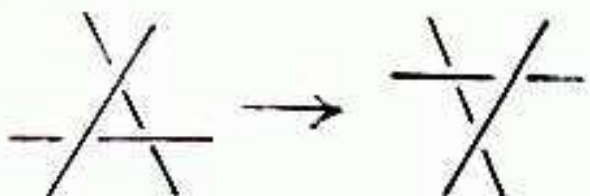
Type II



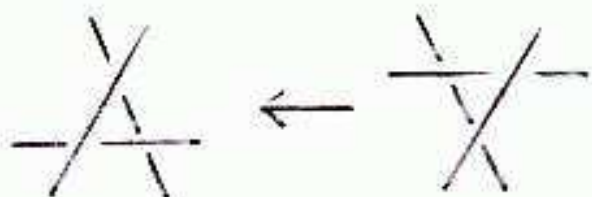
Type II inverse



Type III



Type III inverse



Suppose we are given a long complicated move $K \rightarrow L$.

Imagine taking a film of it, and examining the changes frame by frame.

Each frame will represent either no change from the previous frame or an elementary move.

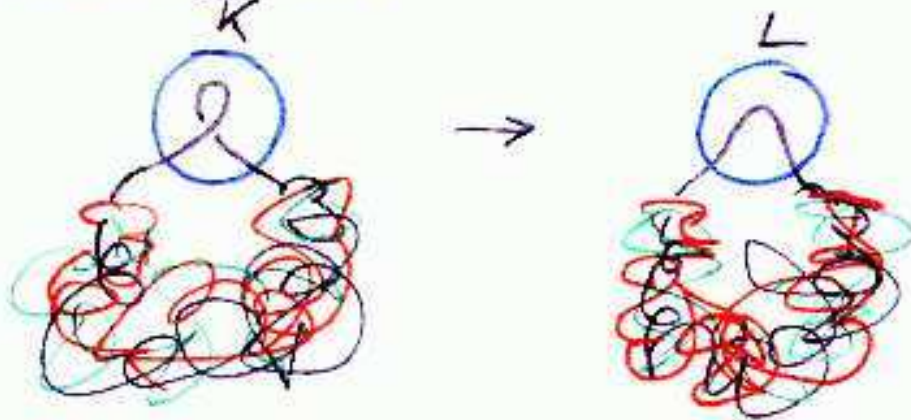
Hence we get a sequence of elementary moves

$$K = K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow \dots \rightarrow K_n = L.$$

Therefore it suffices to prove:

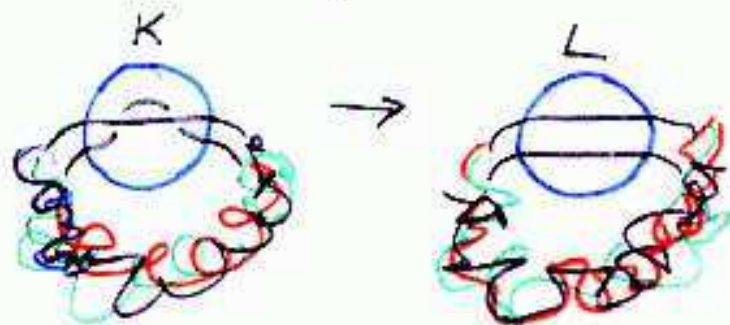
If $\left\{ \begin{array}{l} K \text{ can be 3-colored} \\ K \rightarrow L \text{ is an elementary move} \end{array} \right\}$ then L can be 3-colored.

Type I elementary move.

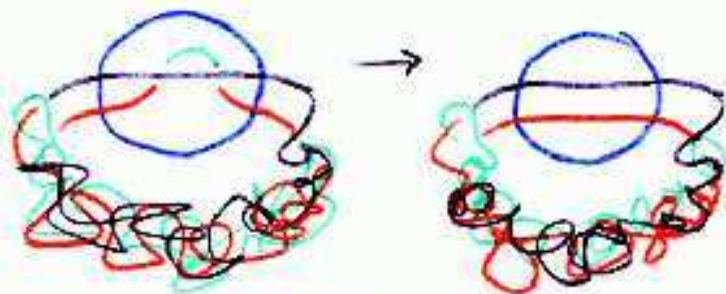


leave the coloring of the rest of the knot unchanged.

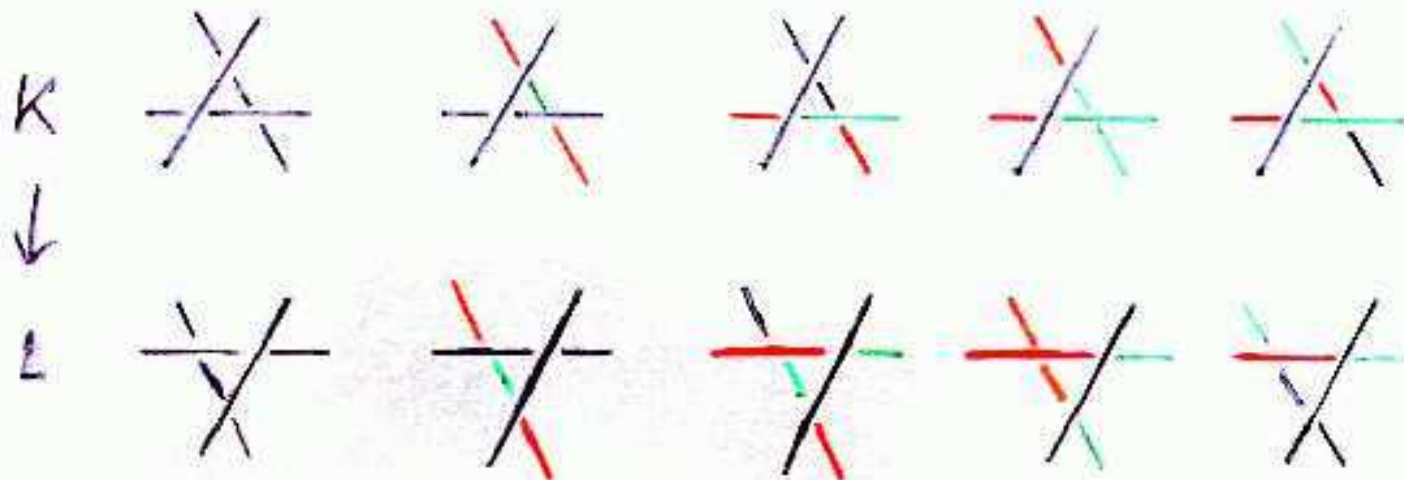
Type II. There are two possibilities.



The inverses are similar



Type III. There are five possibilities



Theorem 3

You can't 3-colour the square-knot.



Proof. Suppose you try.

There are 4 arcs and only 3 colours.

Therefore 2 arcs must be the same colour

But any 2 arcs meet at some crossing.



Therefore the third arc at this crossing must be the same colour by Axiom (2).

Similarly the fourth arc must be the same colour.

Therefore all 4 arcs are the same colour.

This violates Axiom (1).

Therefore you can't 3-colour the square-knot.

Remark You can 3-colour the four-crossing picture of the trefoil.



Corollary Square knot \neq trefoil

Problem How do you prove the square-knot is knotted?

Answer We generalise the invariant.

p-COLOURING

Let p be an odd prime. For example 3, 5, 7, 11, 13, ...

We shall use as "colours" the integers mod p :

$$0, 1, 2, \dots, p-1.$$

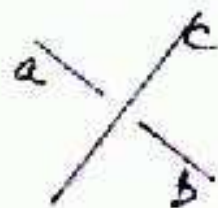
Let K be a picture of a knot.

Definition. We say K can be p-coloured if

Axiom ①. We use at least 2 colours

Axiom ②. At each crossing the overpass is the average of the two underpasses (mod p)

$$a + b \equiv 2c \pmod{p}.$$



Here "equals mod p " means

the two numbers are the same or differ by a multiple of p .

We call p a code of K .

Example $p=3$ This is the same as we had before.

The three "colours" are 0, 1, 2

0 is the average of 1 and 2 because $1+2 \equiv 2 \times 0 \pmod{3}$

1 is the average of 0 and 2 because $0+2 \equiv 2 \times 1 \pmod{3}$

2 is the average of 0 and 1 because $0+1 \equiv 2 \times 2 \pmod{3}$

Therefore there are 1 or 3 colours at a crossing.



Theorem 4. Codes are invariant

Proof. It suffices to show that

if $\left\{ \begin{array}{l} K \text{ can be } p\text{-coloured} \\ K \rightarrow L \text{ is an elementary move} \end{array} \right\}$ then L can also be.

Type I

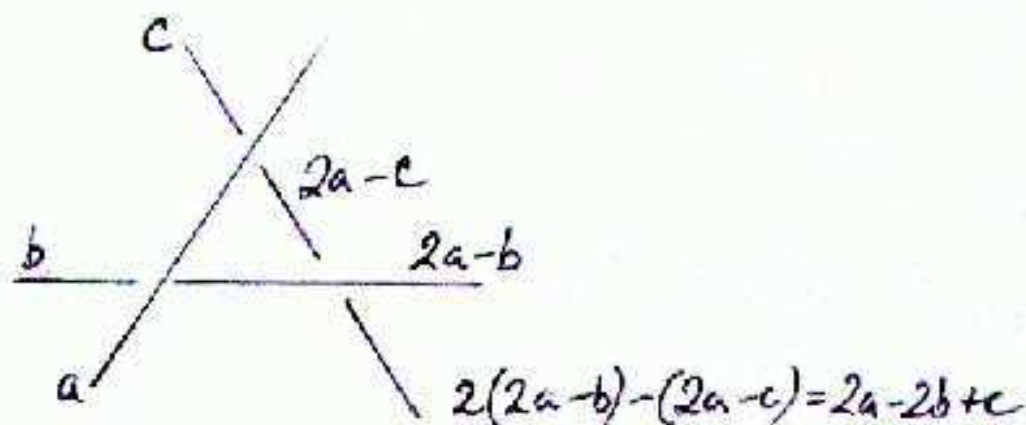


Type II

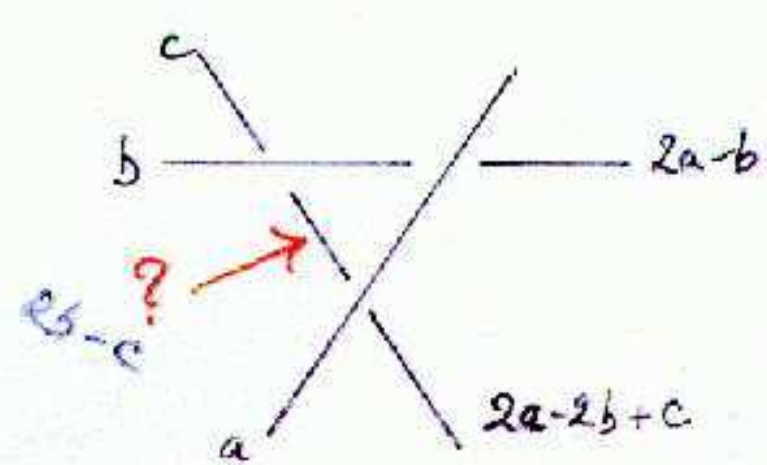



Type III

K , Before

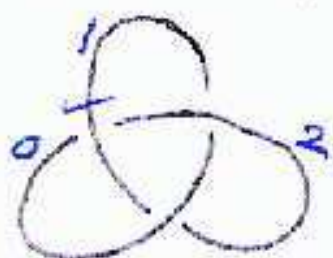


L , After



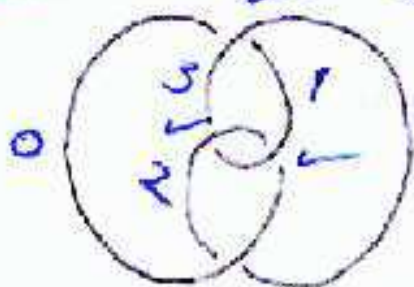
Example 1 The circle has no codes. 

Example 2 The trefoil has code 3.



$$\begin{aligned}0+1 &= 2 \times 2 \pmod{p} \\ 0 &= 3 \pmod{p} \\ p &= 3\end{aligned}$$

Example 3 The square-knot has code 5.



$$\begin{aligned}0+1 &= 2 \times 3 \pmod{p} \\ 0 &= 5 \pmod{p} \\ p &= 5\end{aligned}$$

Therefore the square-knot is knotted.
It is not equal to the trefoil.

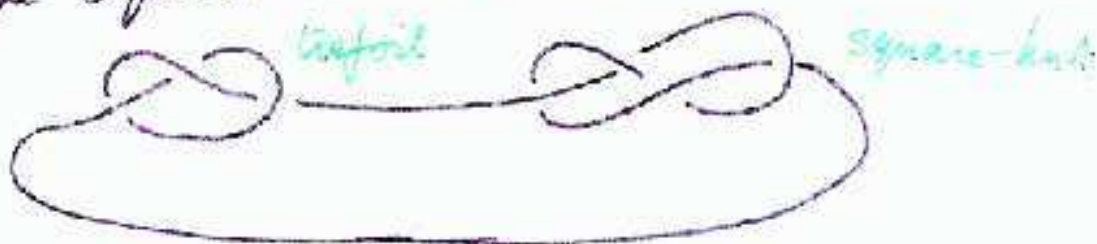
Question Have they any other codes?

Answer No

(because if you can p -down them you can always recolor so that two particular arcs have colors $0, 1$).

Question Can a knot have more than 1 code?

Answer Yes. The product of the trefoil & the square-knot has codes 3 & 5



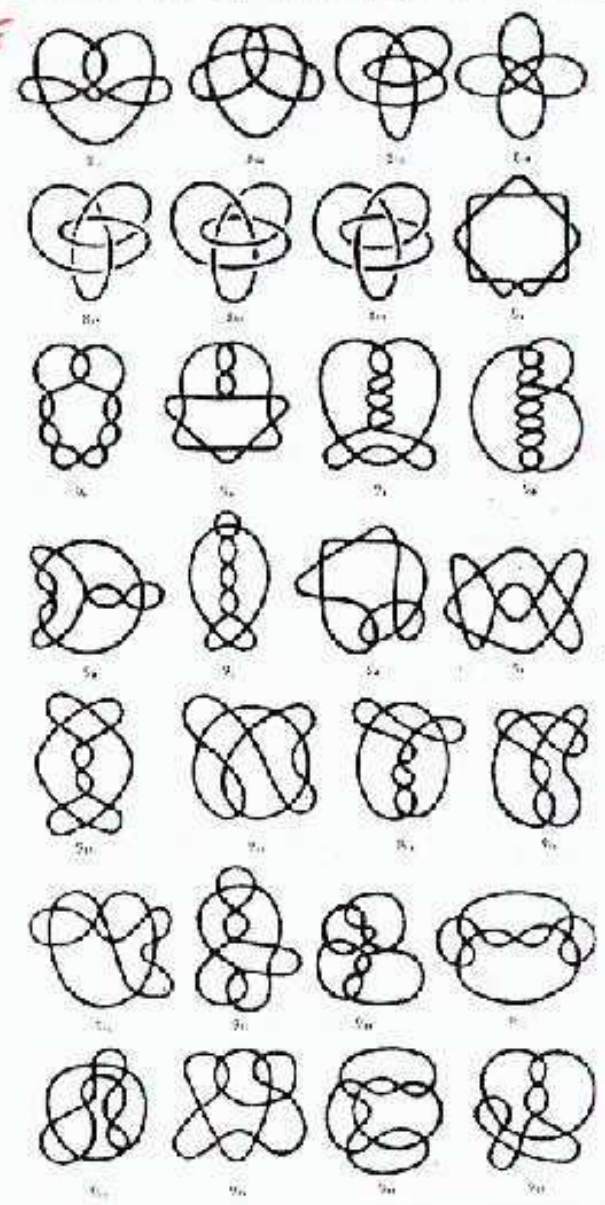
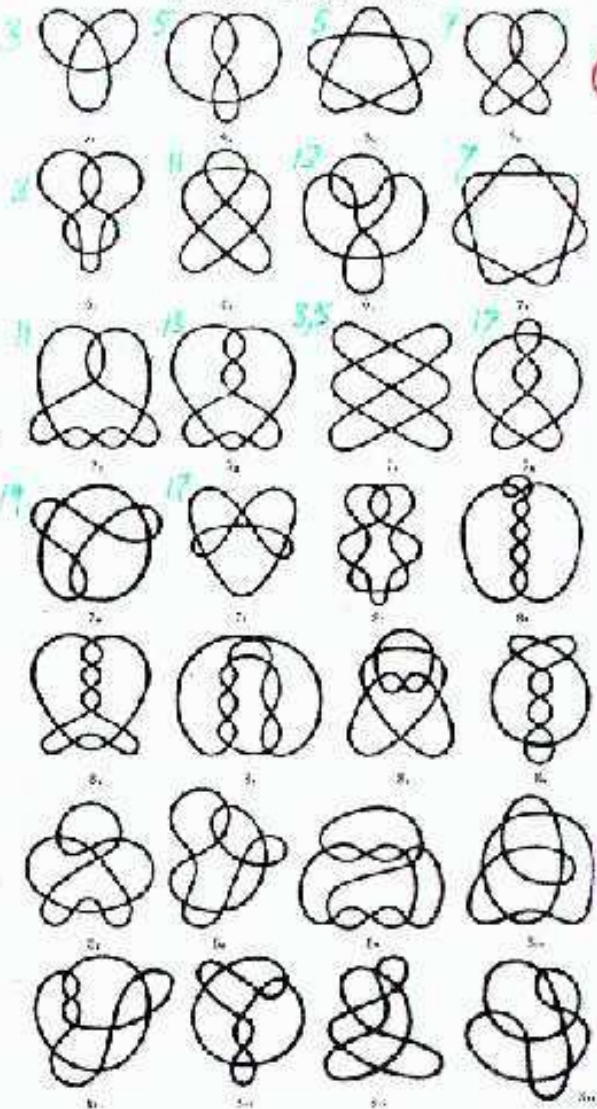
REIDEMEISTER

Knotentabelle

KNOTENTHEORIE

Die Tabelle der folgenden Knotenprobleme ist in neun Überlegungen nach der Arbeit von Reidemeister (1932) unterteilt. Die Knoten sind in Klassen 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000.

1948
(Chelsea)



Theorem 5 Any knot has only a finite number of codes.

Proof

Let K be a picture of a knot.

Going round K label the

arcs a_1, a_2, \dots, a_n

crossings c_1, c_2, \dots, c_n

so that c_i is the front end of a_i .

Define an $n \times n$ matrix M with

rows = crossings c_i

columns = arcs a_j

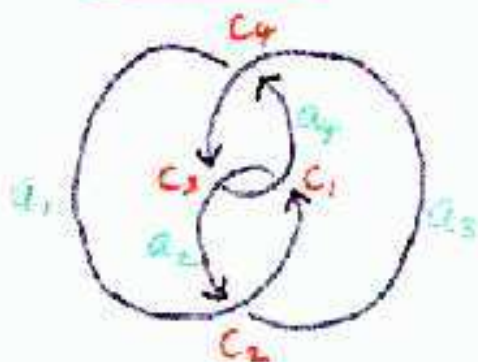
$$M_{ij} = \begin{cases} 1 & \text{if } c_i \text{ has underpass } a_j \\ -2 & \text{if } c_i \text{ has overpass } a_j \\ 0 & \text{otherwise} \end{cases}$$

Define D by crossing out
the last row & column

Let $d = \text{determinant } D$

Then Theorem 5 follows from:

EXAMPLE.



$$M = \begin{matrix} & a_1 & a_2 & a_3 & a_4 \\ \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & -2 \\ -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 \end{bmatrix} \end{matrix}$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\begin{aligned} d &= \begin{vmatrix} 1 & 1 & - & -2 & \\ -2 & 1 & & 0 & 1 \end{vmatrix} \\ &= 3 - (-2) \\ &= 5 \end{aligned}$$

Theorem 6

The codes of K are the prime factors of d .

Proof of Theorem 6

d is odd, because $d \equiv 1 \pmod{2}$

Suppose p is a code of K .

Choose a p -colouring of K .

Let x_i be the colour of a_i

$$\text{Let } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\text{Then } Mx \equiv 0 \pmod{p}$$

We can recolour so that $x_n = 0$
(by subtracting x_n from each colour)

$$\text{Let } y = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$\text{Then } Dy \equiv 0 \pmod{p}$$

But $y \not\equiv 0 \pmod{p}$ by Axiom ①

$$\text{Therefore } d \equiv 0 \pmod{p}$$

$\therefore d$ is a multiple of p .

$\therefore p$ is a prime factor of d .

codes of K are prime factors of d .

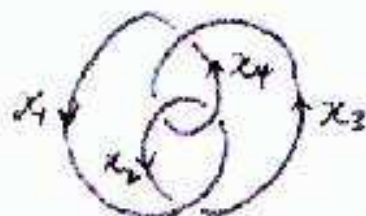
(and conversely)

EXAMPLE

$$D = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\therefore D \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \pmod{2}$$

$$\therefore d \equiv 1 \pmod{2}$$




$$Mx = \begin{pmatrix} 1 & 1 & 0 & -2 \\ -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ = \begin{pmatrix} x_1 + x_2 - 2x_4 \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Dy = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Theorem 7 Knots form a semi-group.

Proof

To each knot  there corresponds a knotted curve



Define the product $A \times B$ of two knotted curves by juxtaposition



Verify that the product is

① Associative $(A \times B) \times C = A \times (B \times C)$



② Commutative $A \times B = B \times A$



③ Has a unit $A \times 1 = A$



Question Do they form a group?

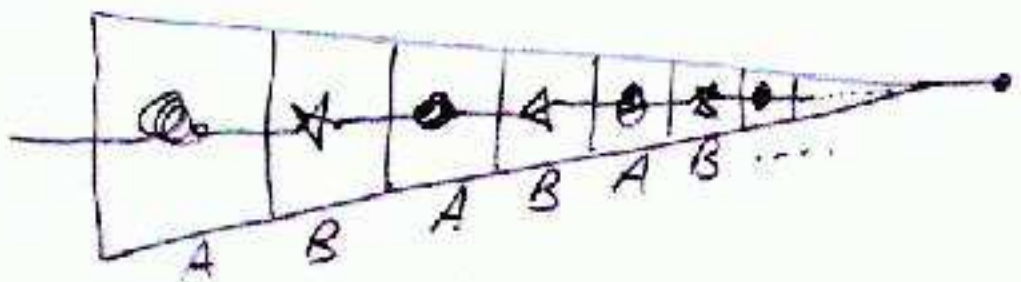
In other words, given A is there an inverse A^{-1} such that $A \times A^{-1} = 1$?

Answer No

Theorem 8 If $A \times B = 1$ then $A = B = 1$.

Proof of Theorem 8 Suppose $A \times B = I$

Let $X = A \times B \times A \times B \times A \times B \times \dots$



$$\begin{aligned} \text{Then } X &= (A \times B) \times (A \times B) \times (A \times B) \times \dots \\ &= I \times I \times I \times \dots \\ &= I \end{aligned}$$

$$\begin{aligned} \text{Also } X &= A \times (B \times A) \times (B \times A) \times \dots \\ &= A \times (A \times B) \times (A \times B) \times \dots \\ &= A \times I \times I \times I \times \dots \\ &= A \end{aligned}$$

Therefore $A = I$

Similarly $B = I$.

Definition

Call a knot P **prime** if $P = A \times B$
implies either $A = 1$ or $B = 1$.

Examples

The trefoil and square-knots are prime.

Theorem 9

Any knot K can be uniquely factored
into primes:

$$K = P_1 \times P_2 \times \dots \times P_n.$$

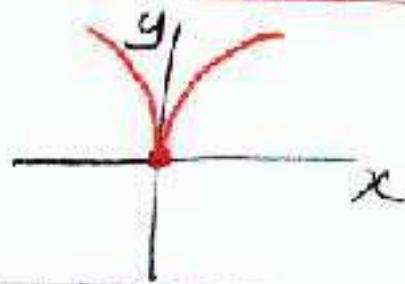
Therefore knots are like positive integers
because they have

products
associativity
commutativity
unit
primes
unique factorization

APPLICATION: SOLUTIONS OF EQUATIONS

Example ① $x^2 = y^3$; $x, y \in \mathbb{R}$

Cusp (differential invariant)



Example ② $x^2 = y^3$; $x, y \in \mathbb{C}$

Solution = surface $M^2 \subset \mathbb{C}^2 = \mathbb{R}^4$



Question What does M look like at the origin?

Answer Take a little ball around the origin, D^4

Take its surface $S^3 = \partial D^4$

Then M^2 meets S^3 in a curve C^1 .

Question What does C^1 look like in S^3 ?

Answer It's a trefoil knot.

$\therefore M^2$ is locally knotted at the origin.

Conclusion A complex cusp is a topological invariant
(much deeper than a real cusp).

Example ③ $x^2 + y^3 + z^5 = 0$; $x, y, z \in \mathbb{C}$

Solution = surface $M^4 \subset \mathbb{C}^3 = \mathbb{R}^6$.

At the origin M is locally knotted with
a knot of S^3 in S^5 .

HIGHER DIMENSIONS




Notation

Define the (n+1)-disk D^{n+1} in \mathbb{R}^{n+1} by



$$x_1^2 + \dots + x_{n+1}^2 \leq 1.$$

Define the n-sphere $S^n = \partial D^{n+1}$, the boundary of D^{n+1} ,
 given by $x_1^2 + \dots + x_{n+1}^2 = 1$

Examples

n	disk D^{n+1}	sphere S^n	
0	$D^1 = \text{interval}$	$S^0 = \text{pair of points}$	
1	$D^2 = \text{disk}$	$S^1 = \text{circle}$	
2	$D^3 = \text{ball (solid)}$	$S^2 = \text{sphere (hollow surface)}$	

Remark

Circles link & knot in \mathbb{R}^3  

Question

In what dimension do spheres link & knot?

	LINKING	KNOTTING	UNKNOTTING
n	is a $2n+1$ phenomenon	is a codimension 2 phenomenon	is a codimension 3 phenomenon
1	Two circles link in \mathbb{R}^3 because $2(1)+1=3$	A circle knots in \mathbb{R}^3 because $1+2=3$	A circle unknots in \mathbb{R}^4 because $1+3=4$
2	Two spheres link in \mathbb{R}^5 because $2(2)+1=5$ Classical	A sphere knots in \mathbb{R}^4 because $2+2=4$ Emil Artin 1924	A sphere unknots in \mathbb{R}^5 because $2+3=5$ Zeeman 1961
50	Two S^{50} link in \mathbb{R}^{101}	An S^{50} knots in \mathbb{R}^{52}	An S^{50} unknots in \mathbb{R}^{53}

LINKING

$$\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$$

Take a circle $S^1 \subset \mathbb{R}^2$

Let D^2 be transversal to S^1

$$\text{Let } S_*^1 = \partial D^2$$

Then S_*^1 links S^1 in \mathbb{R}^3

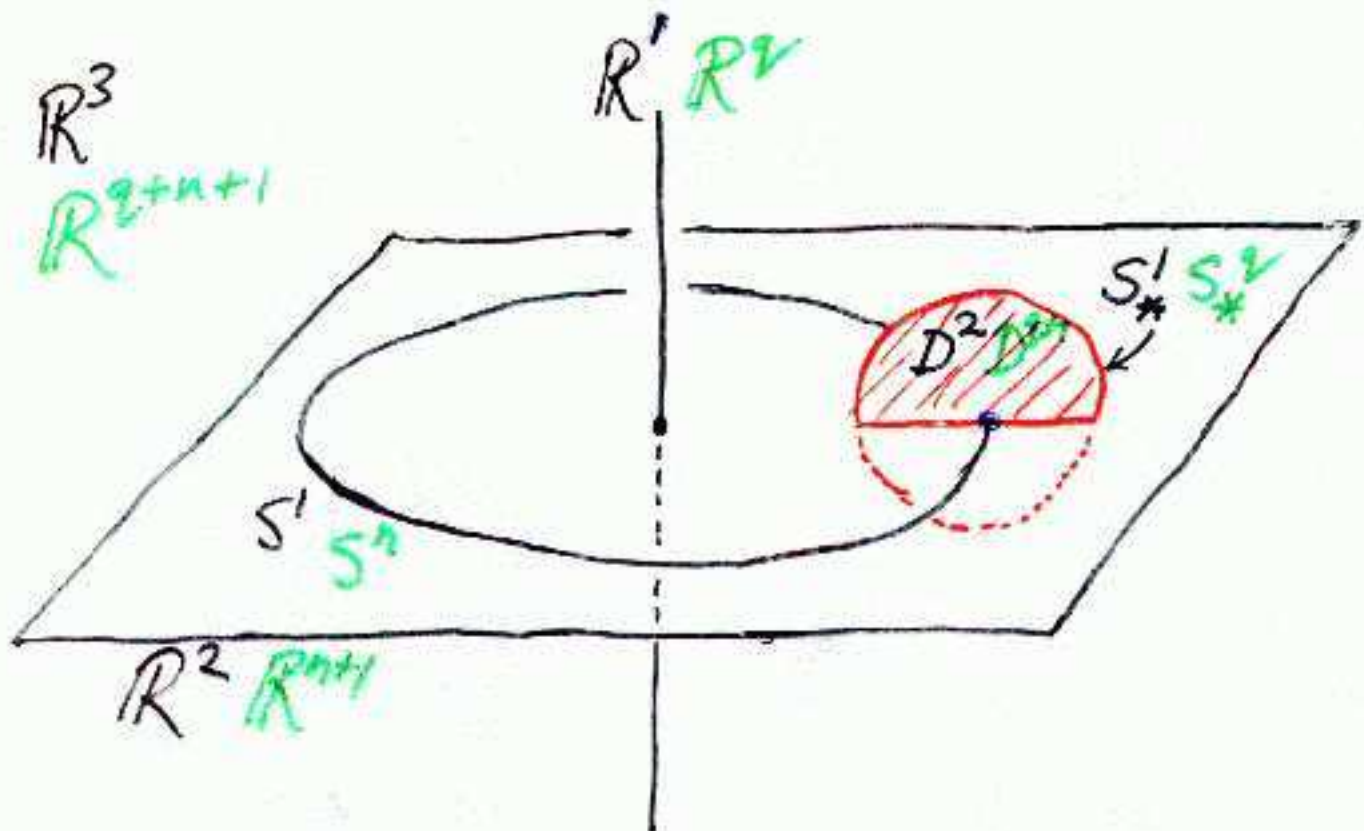
$$\mathbb{R}^{2+n+1} = \mathbb{R}^2 \times \mathbb{R}^{n+1}$$

Take a sphere $S^n \subset \mathbb{R}^{n+1}$

Let D^{2+1} be transversal to S^n

$$\text{Let } S_*^n = \partial D^{2+1}$$

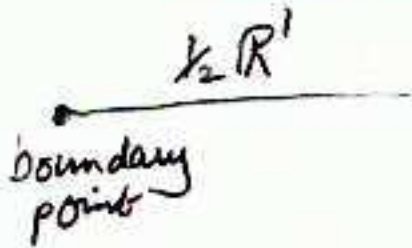
Then S_*^n links S^n in \mathbb{R}^{2+n+1}



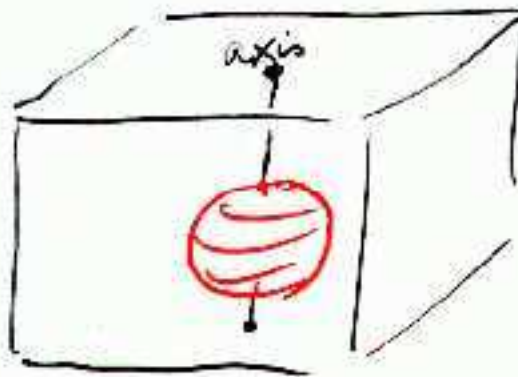
Example

S_*^2 links S^2 in \mathbb{R}^5

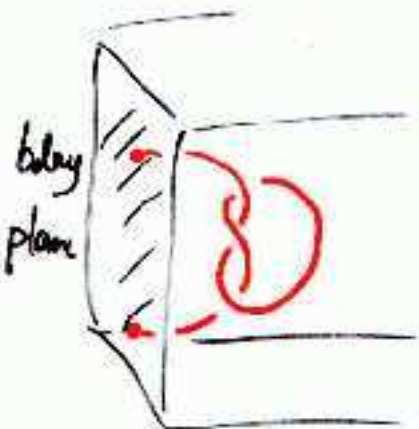
KNOTTED SPHERE IN \mathbb{R}^4



$\text{spin } \frac{1}{2}\mathbb{R}^1 = \mathbb{R}^2$

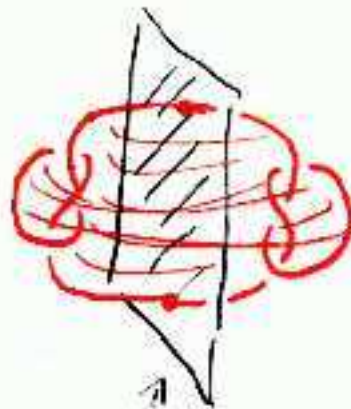


$\text{spin } \frac{1}{2}\mathbb{R}^2 = \mathbb{R}^3$
 \cup
 $\text{spin } A = S^2$



$\frac{1}{2}\mathbb{R}^3$
 \cup

Knotted arc K



2-dimensional axis

$\text{spin } \frac{1}{2}\mathbb{R}^3 = \mathbb{R}^4$
 \cup
 $\text{spin } K = \text{knotted } S^2$

THEOREM Any S^2 in \mathbb{R}^5 is unknotted.

Proof Given $S^2 \subset \mathbb{R}^5$.

Let V be a vertex in general position

From dimensional considerations at most a finite number of rays through V meet S^2 twice, & none thrice.

\therefore each such ray meets S^2 in a near point N & a far point F .

Let A be a disk in S^2 enclosing all near points and no far points.

Let $B = \text{closure}(S^2 - A)$

The cone VA is non-singular, because it contains no far points.

$$\therefore S^2 = A \cup B$$

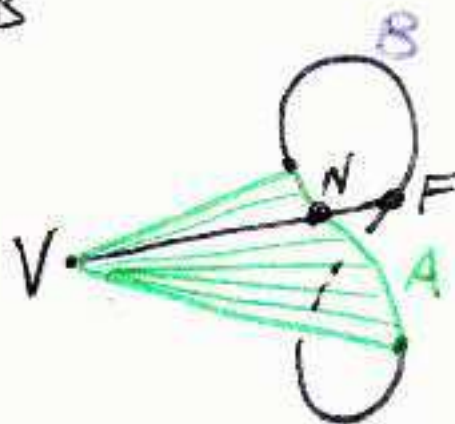
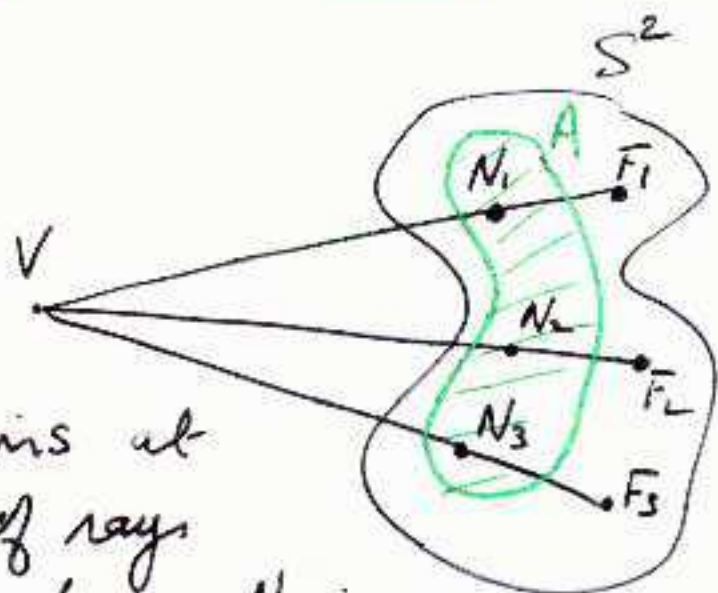
$$\sim V(\partial A) \cup B, \text{ sliding } A \text{ across } VA \text{ to } V(\partial A)$$

$$= V(\partial B) \cup B, \text{ since } \partial A = \partial B$$

$$= \partial(VB)$$

$$= \text{unknotted because}$$

it bounds the disk VB .



DIMENSIONAL CONSIDERATIONS

Lemma

In \mathbb{R}^3 , given two lines and a vertex V in general position, then \exists unique ray through V meeting both lines.

Proof

Let L, M be the lines

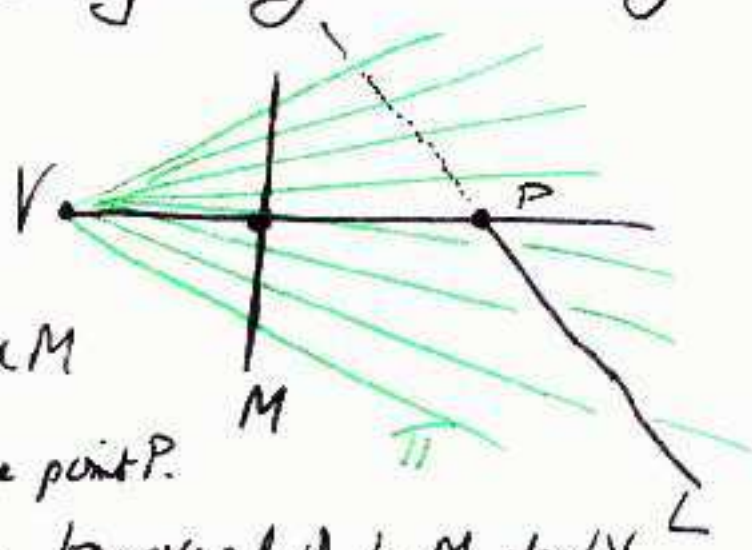
Let $\Pi =$ plane containing V, M

Then L pierces Π at a unique point P .

Therefore VP is the unique transversal of L, M through V .

Corollary 1 \exists ray through V meeting 3 given lines.

Corollary 2 A projection of a knotted curve in \mathbb{R}^3 has generically a finite number of crossings & no triple points.



Lemma In \mathbb{R}^5 , given two planes and a vertex V in general position, then \exists unique ray through V meeting both planes.

Proof as above.

Corollary 1 \exists ray through V meeting 3 given planes.

Corollary 2 The projection of a knotted sphere in \mathbb{R}^5 from V has only a finite number of crossings & no triple points.