ROYAL INSTITUTION MATHEMATICS MASTERCLASS

Gyroscopes and Boomerangs

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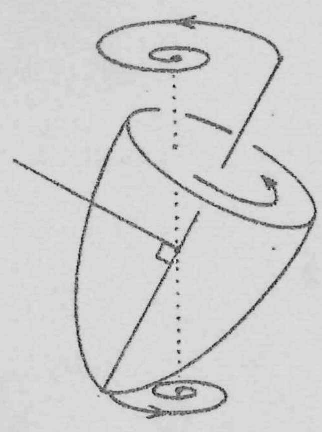
This is the book accompanying the video



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Acknowledgements

I developed this approach to gyroscopes gradually over ten years while giving Mathematics Masterclasses to 13 year olds at the Royal Institution and the University of Warwick. I should like to thank those two institutions for providing me with the opportunity, and the various industrial firms who sponsored the classes. I should also like to thank all the young people who came to them and shared their love of mathematics with me. I remember one young lady who whispered to me "it's lovely to be amongst people who don't think you're odd". I particularly thank Shahnaz Akhtar and Timothy Wright from the Warwick class who were my two assistants in the video. I am indebted to the Department of Education and Science for providing the financial support to make the video, and for distributing it to local authorities for free copying by all schools. I am grateful to John Jaworski and all his team in the BBC Open University Production Unit for making the video. I should like to thank Denise Roby for typing the book.

Introduction

This is the book accompanying the video *Gyroscopes and Boomerangs*. It is the second in the series of Royal Institution Mathematics Masterclass Videos. The first was on *Geometry and Perspective*, and this one follows the same pattern. The idea is to provide enrichment material for gifted 13-year olds. The subjects were chosen for their interest and importance, and to complement the school mathematics syllabus.

This video is about mechanics. Developing an intuition about mechanics is as important as becoming literate and numerate. For in today's world we all have to use machines of one sort or another, and an intuition about mechanics enables us to handle them more efficiently, more sympathetically and more enjoyably.

Unfortunately in some schools the teaching of mechanics has been dwindling due to the rise of computing and statistics. This is a great pity because mechanics is the cornerstone of physics and engineering, and has been one of the main driving forces behind the creation of mathematics ever since Newton discovered the laws of motion in 1666.

Mechanics also provides an ideal illustration of the scientific method. First you observe something interesting and make a mathematical model of it. Then you work out the formulae, put in the measurements, make the predictions, and finally test them by experiment. In mechanics the observations are intriguing, the formulae are non-trivial, the measurements are straightforward, the predictions are precise, the experiments are easy to do needing only the simplest of equipment, and the predictions can be confirmed with surprising accuracy. There is no better way to learn science.

So which part of mechanics should we start with? My choice was to go for spinning motion because it is the more surprising. Gyros are fascinating, and discovering why they work can be exciting and deeply satisfying. Their behaviour can be described by a simple gyro law that explains why they precess, which way they precess, and why they go on precessing. The law also explains the behaviour of tops and boomerangs, and the precessions of the earth's axis and the moon's orbit. There is no need to use calculus: one can develop the theory of precession using only elementary mathematics, and work out a formula for the precession time that is easy to test.

I hope this video and book may appeal to people of all ages, but I particularly address myself to young students. If you are about 12 or 13 you will have reached the age when you can think abstractly, and can begin to appreciate the subtleties of motion. I hope you will be tempted to repeat the experiments for yourself and to make your own boomerang.

How to use this book

The video is made in three parts:

Part 1: The Gyro Law (30 minutes)
Part 2: The precession time (15 minutes)
Part 3: Tops, eggs and boomerangs (15 minutes).

The book contains notes summarising each part of the video. Also, more importantly, it contains worksheets to do after seeing each part and before seeing the next part. If you can tackle and solve all the problems on each of the first two worksheets you will not only have a better understanding of the theory, but you will also be able to appreciate the next part of the video much more deeply. The last two worksheets provide an opportunity to reinforce what has been learnt, and to have a go at some harder problems. The solutions to the problems on all the worksheets are given at the end of the book.

If you are a teacher using the video to run a class I recommend showing one part at a time. After each part photocopy the worksheet for that part and hand it out as an exercise for your students. You may also want to photocopy the notes for them to have as a reference. When they have had a chance to try the problems you can use the solutions to help them with the ones they have not been able to solve.

If you are a young student using the video for private study I urge you to follow the same pattern (and not to cheat!). You will get far more out of it if you do things in the right order. Discipline yourself to switch off the video after watching Part 1, and have a go at Worksheet 1 before watching Part 2, because this will enable you to calculate and predict the results of the experiments in Part 2: then it will be more exciting to watch and see if your predictions are correct.

Similarly if you do Worksheet 2 before watching Part 3 you will have a chance to discover for yourself how tops work before seeing them in motion, and in slow motion. Anything you discover for yourself you will never forget, especially when you then see it confirmed with your own eyes.

Worksheet 3 is for you to do at the end of the video and explains how to make your own boomerang. Worksheet 4 is a bit harder and gives the proof of why boomerangs work.

Finally there are two appendices for those of you who are curious and want to go into the subject more deeply. Appendix 1 proves that Newton's law of linear motion implies the analogous law for spinning motion, and gives a simple introduction to calculus. Appendix 2 proves why a spinning egg stands on end.

Gyroscopes and boomerangs

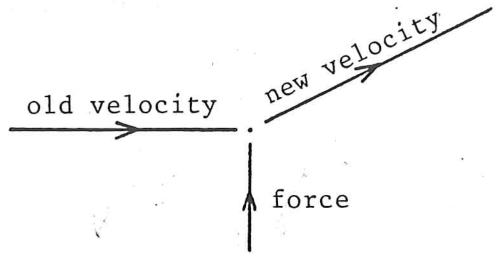
Notes on the Video Part 1: The Gyro Law

A *gyro* is anything that is symmetrical about an axis and is spinning about that axis. Examples of gyros include wheels, tops, drawing pins, eggs, little gyros that you can buy in toy shops, precision gyros in navigational instruments, and large gyros in ship stabilisers. At first sight gyros seem to behave in a mysterious way, and the aim of this video is to unravel those mysteries and explain their behaviour in terms of Newton's law of motion.

In the first experiment if you hold the axis of a spinning gyro and try to move it so that it points in another direction then you will find it will resist your attempt and will tend to go off in a different direction; it seems to develop a life of its own. In the second experiment if a gyro is hung off centre then the axis will slowly rotate about the point of suspension; this slow rotation is called *precession.* Before we can explain these examples of spinning motion, however, we shall start with the much simpler case of linear motion.

Linear motion

If we apply a force to a moving object then this will cause a change in velocity.



Question: If we know the old velocity and the force how do we predict the new velocity?

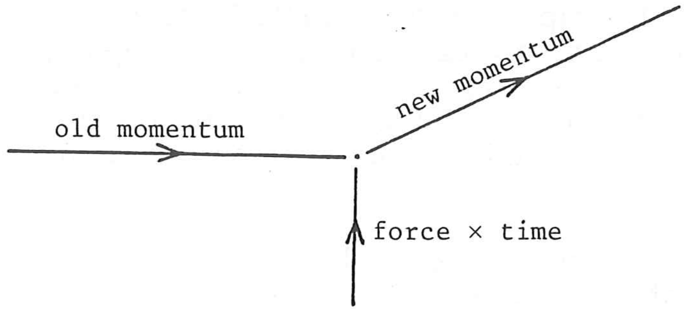
Answer: It depends on (1) mass, (2) time and (3) vectors.

(1) **Mass.** The mass of the object is important because the larger the mass the smaller the change in velocity. We take account of mass by introducing the concept of *momentum*, which is defined:

$$\text{momentum} = \text{mass} \times \text{velocity}.$$

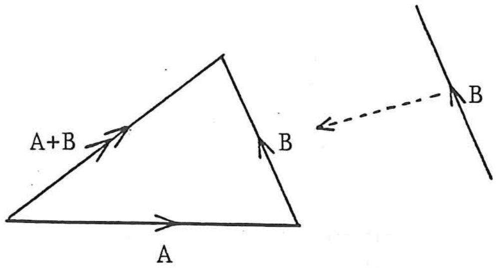
Intuitively momentum measures the reluctance of the moving object to be deflected from its path. We can then replace the velocity arrows by momentum arrows in the same directions.

(2) Time. The length of time that the force is applied is important because the longer the time the greater change in velocity. We take account of time by replacing the force arrow by a "force × time" arrow in the same direction.

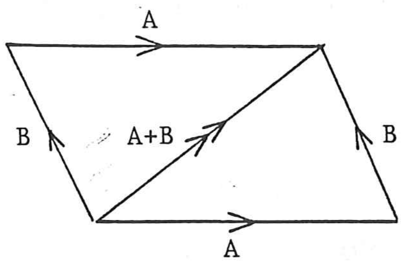


(3) Vectors. Anything that has size and direction is called a *vector*. For example velocity is a vector, because the size of velocity is the speed in centimetres per second and the direction of velocity is the direction of motion. A vector can be represented by an arrow, or equally well by any other parallel arrow of the same size. All the arrows in the two diagrams above represent vectors.

Given two vectors A and B define their *sum* A+B as follows: move B to fit on the end of A, and define A+B to be the vector going from the beginning of A to the end of B.

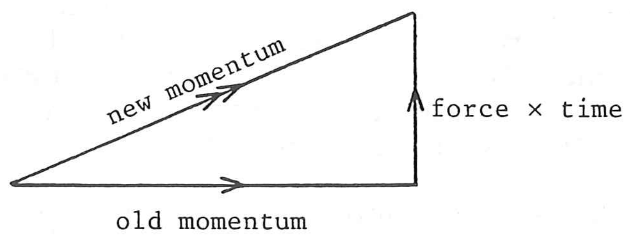


An alternative method is to construct a parallelogram with sides equal to A and B, and then the diagonal gives A+B. This method is called the parallelogram rule.



Newton's law of linear motion

$$\text{New momentum} = (\text{old momentum}) + (\text{force} \times \text{time})$$



This enables us to predict the new momentum, and hence the new velocity, giving the answer to the question at the beginning.

Remark 1: We have drawn the force at right-angles to the momentum although of course it could be at any angle.

Remark 2: The usual version of Newton's law involves acceleration, but we choose this simpler version here in order to avoid having to worry about acceleration. The two versions are shown to be equivalent in Appendix 1.

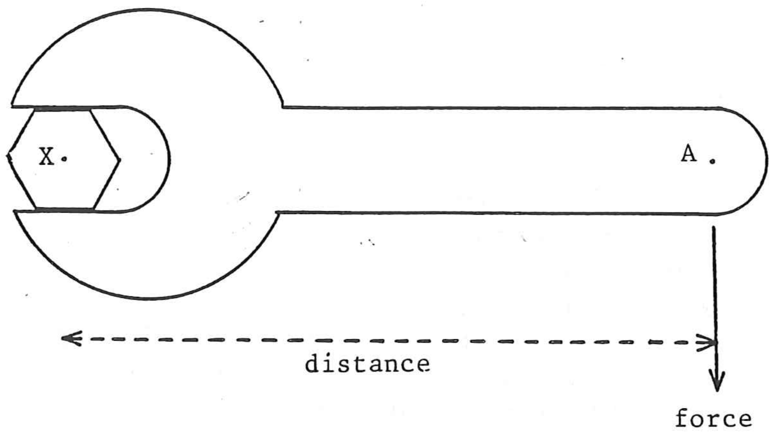
Spinning motion

We now want to introduce a spinning version of Newton's Law. This will be done by replacing the linear words *force, velocity* and *momentum* by the corresponding spinning words *torque*, *spin* and *angular momentum.* The first thing to do is to explain the meaning of those words.

Torque

Torque means "twisting force", and we introduce it in three steps going from the concrete to the more abstract, as follows.

Step 1 Imagine using a spanner to tighten a nut X by pushing down with a force at the point A .

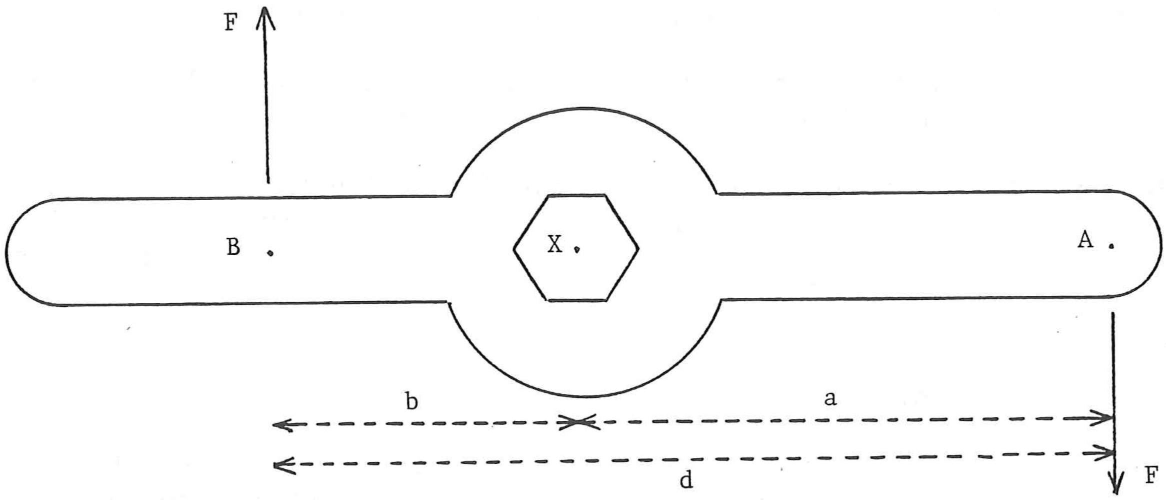


Define:

$$\textit{torque about } X = \text{distance} \times \text{force}$$

Notice that the greater the force the greater the torque, and the greater the leverage the greater the torque. Define the *torque-axis* to be the direction in which the nut will move, which is perpendicular to the paper away from you. Here we are assuming that the nut has a right-hand thread, and so this is called the *right-hand screw rule.*

Step 2 Imagine using a double spanner to tighten the nut, by pushing down at A with force F and up at B with an equal and opposite force.



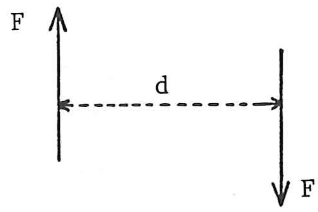
Then

$$\text{torque about } X = aF + bF = (a + b)F = dF\ .$$

Notice that the torque dF does not depend upon the position of X , and so in this case (of equal and opposite forces) we can omit the words "about X".

Step 3 Forget about spanners. Given two equal and opposite forces F whose lines of action are a distance d apart, define

$$\begin{aligned} \text{torque} &= \text{distance} \times \text{force} \\ &= dF. \end{aligned}$$



To find the direction of the torque axis, first identify the plane containing the two forces (here the plane of the paper), then take a line perpendicular to that plane, and finally choose the direction along that line given by the right hand screw rule (namely the direction in which a nut or a corkscrew would move if the torque were applied to it). This procedure for finding the direction of the torque axis is quite important because in applications you will be having to think in three dimensions, which can be confusing if you are not careful.

Summarising, torque is a vector because it has both size (distance × force) and direction (the torque axis).

Spin

The *spin* of a wheel or gyro is measured in revolutions per second. The *spin axis* points along the axis of the wheel in the direction given by the right-hand screw rule:

Spin is a vector because it has both size (revs/sec) and direction (the spin axis).

Angular momentum

Angular momentum is a vector in the same* direction as the spin axis. To calculate the angular momentum imagine dividing the wheel into pieces of different radius; calculate the momentum of each piece as it spins round and round, multiply by its radius, and then add up all the pieces. Notice that the more the mass is distributed towards the rim of the wheel the greater the angular momentum. Intuitively angular momentum measures the reluctance of the axis of the wheel to be deflected (just as in linear motion the momentum of a moving object measures its reluctance to be deflected from its path of motion). Notice the beautiful symmetry of the spinning definitions in terms of the related linear concepts:

$$\text{torque} = \text{distance} \times \text{force}$$
$$\text{angular momentum} = \text{distance} \times \text{momentum}.$$

We can now derive the spinning version of Newton's law by merely replacing linear words by spinning words.

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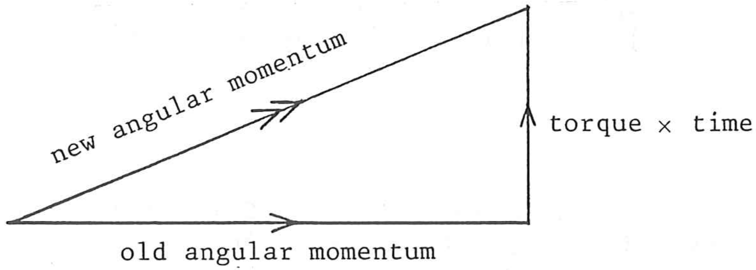
* This is true when the spin axis is along the axis of symmetry, but only approximately true otherwise (see Appendix 1). For example one of Saturn's moons, Hyperion, is very asymmetrical and consequently tumbles chaotically instead of spinning regularly.

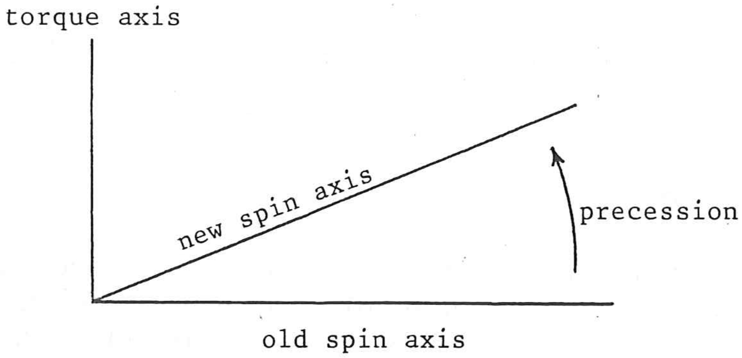
Newton's law of spinning motion

New angular momentum = (old angular momentum) + (torque × time)



Here we only *state* the spinning law, but in Appendix 1 we show how to *prove* the spinning law from the linear law. Therefore from the point of view of developing the theory of motion Newton only needed to postulate the linear law.

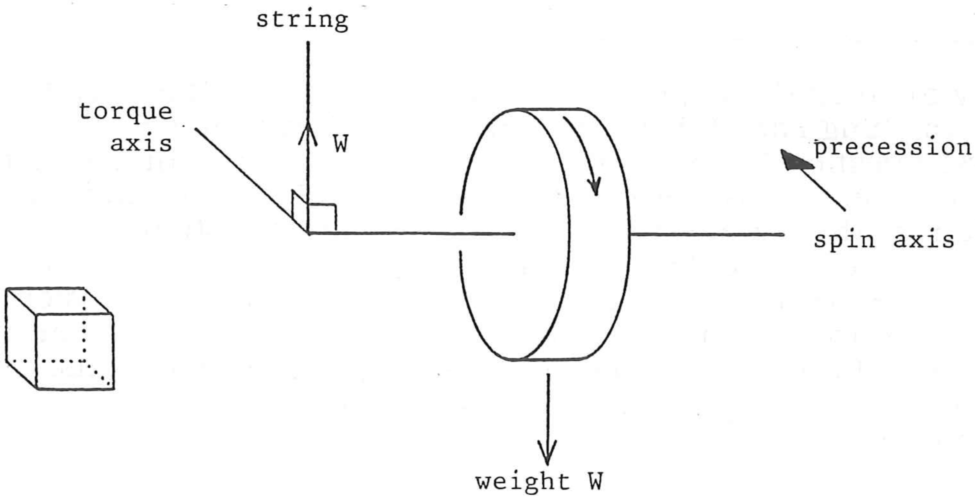
Since the torque-time vector is in the direction of the torque axis, and the angular momentum vector is in the direction of the spin axis, we can draw the directions of the axes as follows:



The picture explains why precession occurs, and which way the spin axis will precess. We can summarise it in a beautiful little gyro law, as follows.

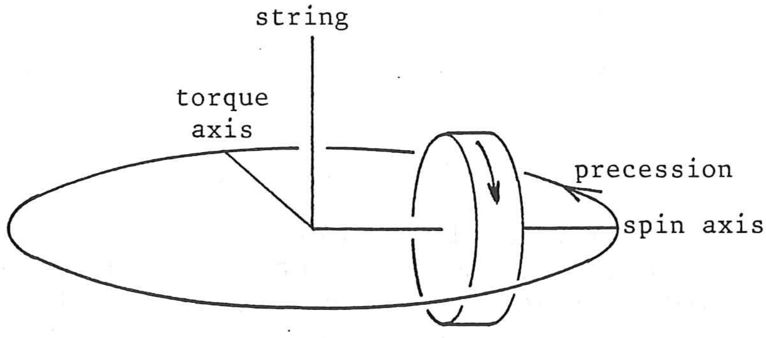
Gyro Law
The spin axis chases the torque axis.

We can now apply this to the spinning bicycle wheel hanging on a string.



The two forces acting on the wheel are the weight W downwards and the equal and opposite pull upwards of the string. To work out the direction of the torque axis first identify the plane containing the two forces, which is the plane of the paper; then take a line perpendicular to the paper in the direction given by the right-hand screw rule, which is away from you. (This is indicated in the diagram by a line drawn parallel to an edge of the little cube in perspective.)

By the gyro law the spin axis chases the torque axis and so begins to precess horizontally away from you. This in turn causes the torque axis also to rotate horizontally, always keeping 90° ahead of the spin axis. By the gyro law the spin axis chases the torque axis round and round, causing the steady precession.



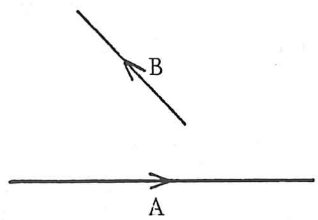
Gyroscopes and Boomerangs

Worksheet 1

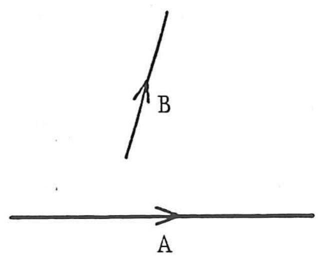
Try to do all these questions after you have seen the video Part 1, and before watching Part 2, because it will make Part 2 much easier to follow. The first question is about adding vectors. The next three questions are about torque, to give you practice in calculating it and finding the direction of the torque axis. The rest of the questions are about the spinning bicycle wheel hanging on a string. The most important question is number 8 because that gives the formula for the precession time. I shall prove this formula in the next part of the video, and do the experiments to test the predictions made by your calculations. So it is a very good idea if you can do the calculations before seeing the experiments.

1. In each case measure the lengths of the vectors A and B in centimetres. Draw a parallelogram with sides parallel and equal to A and B . Draw in the diagonal A+B , and measure its length.

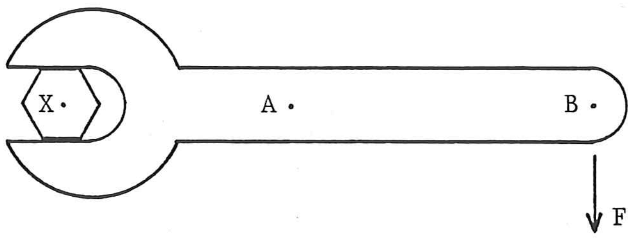
(i)



(ii)



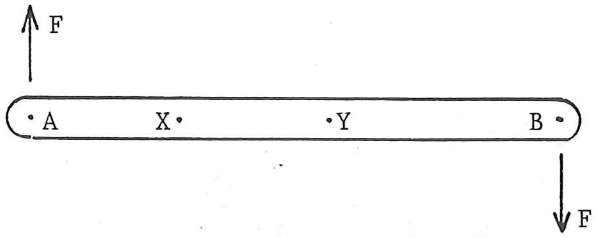
2. Measure XB in centimetres.



If you push down at B with force F what is the torque about X ? The torque axis is perpendicular to the page; use the right-hand screw rule to determine the direction of the torque axis, whether it is towards you or away from you.

If you were to push down at A with force 2F would this produce a larger or smaller torque about X ?

3. Measure the distances AX , XB and AB .

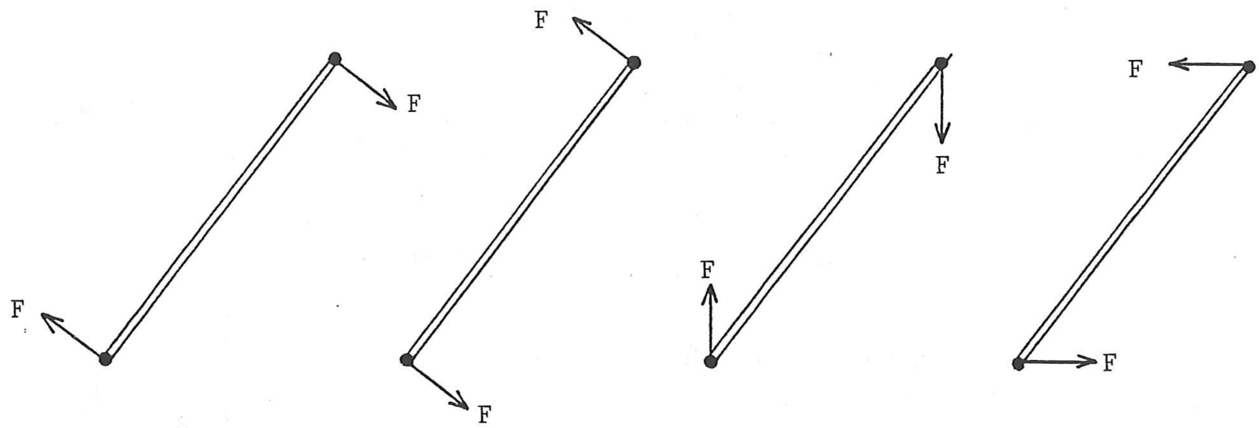


Calculate the torque about X when you

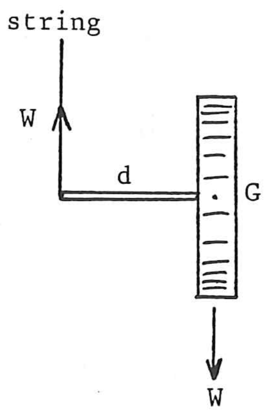
(i) push up at A with force F ,
(ii) push down at B with force F ,
(iii) do both at once.

What is the direction of the torque-axis? Calculate the torque about Y in each case, and verify that in case (iii) the answer is the same as that for X .

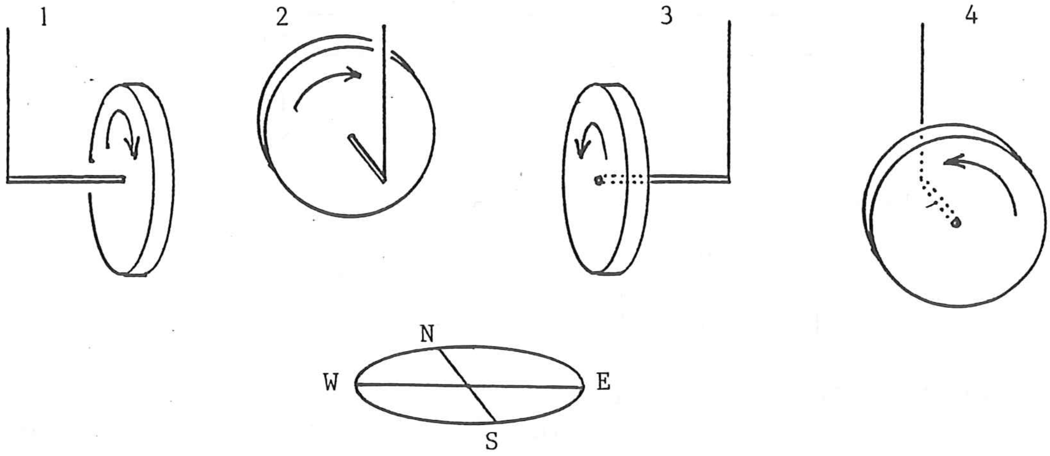
4. In each of the four cases measure the distance between the lines of action of the two forces, calculate the torque, and use the right-hand screw rule to determine whether the torque axis is towards you or away from you.



5. The wheel with centre of gravity G is hanging on a string a horizontal distance d from G . The two forces on the wheel are the weight W downwards at G , and an equal and opposite pull upwards by the string. What is the torque, and in which direction is the torque axis?



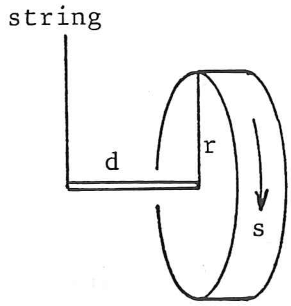
6. The diagram shows four positions of a spinning wheel hanging on a string, and a horizontal circle giving the compass directions North, South, East and West.



Use the right-hand screw rule to determine the compass directions of the spin-axis and torque axis in each of the four positions. State the gyro law. Which way round the horizontal circle will the spinning wheel precess?

7. A wheel of radius r centimetres is spinning at s revolutions per second. How far does a point on the rim travel
   (i) in one revolution?
   (ii) in one second?
   What is the speed of a point in the rim in cm/sec? Calculate the speed to the nearest cm/sec when $r = 24.5$ and $s = 3$.

8. A spinning bicycle wheel is hanging from a string as shown. Let
   r = radius of wheel in cm
   d = distance of string from wheel in cm
   s = spin in revs/sec
   g = gravity = 1000 cm/sec/sec
   t = time to precess once round in sec.



The formula for t is given by $t = \dfrac{4\pi^2 r^2 s}{gd}$.

Given that $r = 24.5$ calculate t to the nearest second in the three cases:
   (i) $s = 3$, $d = 6.5$
   (ii) $s = 2$, $d = 6.5$
   (iii) $s = 3$, $d = 15$.

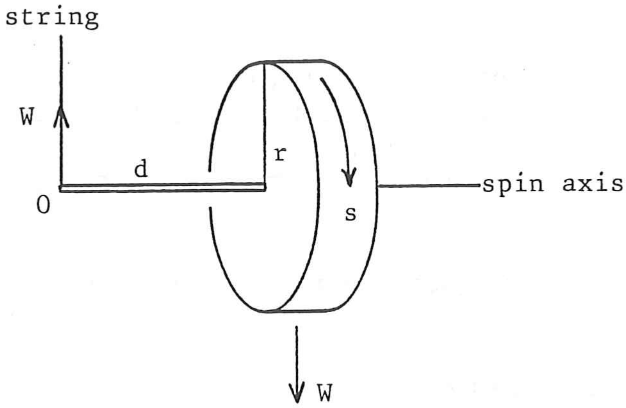
9. If you use the more accurate figure for gravity of $g = 981$ cm/sec/sec what difference would this make to your answers in Question 8?

Gyroscopes and boomerangs

Notes on the Video Part 2: The precession time

The aim of the video Part 2 is to prove the formula for the precession time of a spinning wheel, and to do experiments to test this formula. You will have already used the formula in Worksheet 1 Question 8 to work out the time in three cases, and your answers will be the predictions for the experiments.

Consider a wheel spinning with 0 fixed and spin axis horizontal:



Let r = radius of wheel in centimetres
d = distance of string from wheel in centimetres
s = spin in revolutions per second
g = gravity in centimetres per second per second
t = time to precess once round in seconds.

Then the formula for the precession time is

$$\boxed{t = \frac{4\pi^2 r^2 s}{gd}}.$$

To prove the formula the first thing we need to do is to work out the angular momentum and the torque.

Angular momentum

Let m be the mass of the wheel. There is no need to specify in what units we are measuring the mass because m does not appear in the final formula. For simplicity of calculation let us assume that all the mass of the bicycle wheel is concentrated in the rim. This is not a bad approximation because the spokes are very light, and the hub contributes relatively little to the angular momentum because it is so close to the axis.

The circumference of the wheel is $2\pi r$ cm. Therefore a point on the rim travels $2\pi r$ cm in one revolution and $2\pi rs$ cm in one second, because the spin is s revs/sec. Therefore the speed of the rim is $2\pi rs$ cm/sec.

Therefore the momentum of the rim as it goes round and round is given by:

$$\text{momentum} = \text{mass} \times \text{speed} = m \times 2\pi rs = 2\pi rsm.$$

To obtain the angular momentum we multiply by the distance r of the rim from the axis:

$$\begin{aligned}\text{angular momentum} &= \text{distance} \times \text{momentum} \\ &= r \times 2\pi rsm \\ &= 2\pi r^2 sm.\end{aligned}$$

Torque

The two forces on the wheel are the weight W downwards and an equal and opposite pull of the string upwards. It must be equal because if it were greater then the wheel would fly upwards, and if it were less the wheel would fall downwards. The two forces are a distance d apart, and so

$$\text{torque} = \text{distance} \times \text{weight} = dW.$$

Weight, mass and gravity

The relationship between weight and mass is given by the formula:

$$\text{weight} = \text{mass} \times \text{gravity}.$$

In symbols

$$W = mg.$$

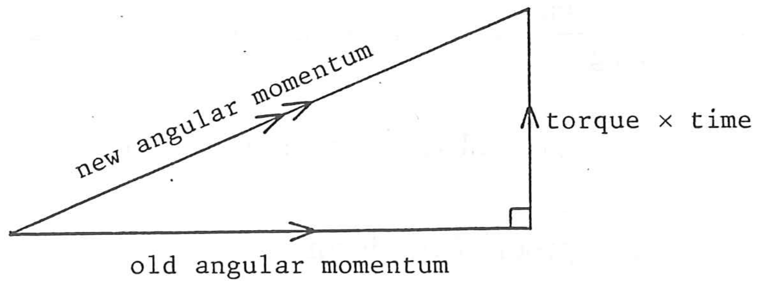
For example if you were on the moon your mass would be the same but the moon's gravity is only one sixth of that on the earth, and so your weight would be only one sixth of your weight on earth. If you were in a space ship with the engines turned off your mass would still be the same but there would appear to be no gravity (because the space ship would be in free orbit with respect to the combined gravitational attractions of earth, sun and moon) and so you would appear to have no weight.

On the earth all objects fall downwards with a constant acceleration which is used to define the earth's gravity g . We need to express g in terms of the same units that we have used for the other measurements, namely centimetres and seconds, and for our calculations a sufficiently good approximation will be g = 1000 cm/sec/sec. A more accurate estimate would be g = 981 cm/sec/sec, but there is no point in using the greater accuracy since the other measurements are only accurate to within about 10%, and so it would not affect the predictions (see Worksheet 1 Question 9). Substituting W = mg in the expression for torque gives:

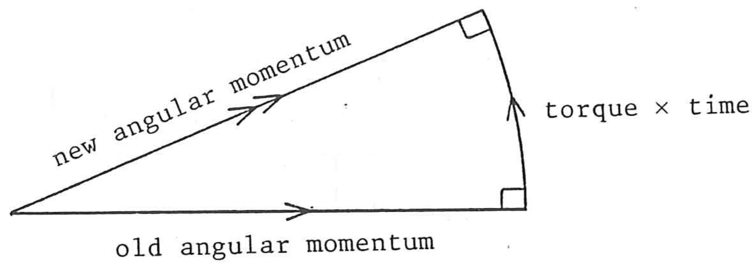
$$\text{torque} = dW = dmg.$$

Proof of the Formula

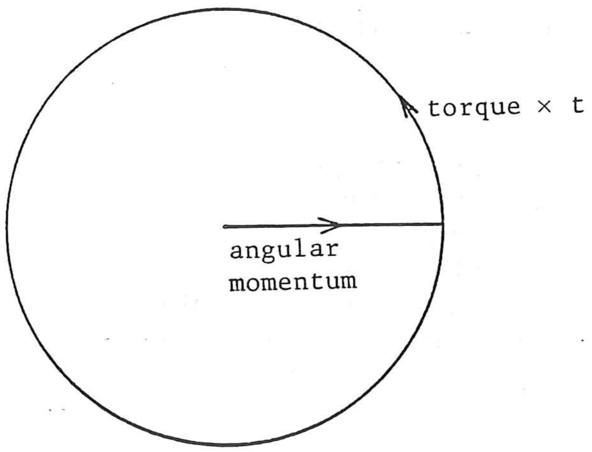
Start with Newton's law for spinning motion, looking down on the wheel from above:



The torque, however, is not in a constant direction because it precesses at the same rate as the spin axis, always keeping 90° ahead. Therefore the right-angle triangle becomes a sector of a circle, with the torque-time path becoming an arc of the circle, always at right-angles to the radius, which represents the angular momentum.



In time t the spin axis precesses round a whole circle.



Since the circumference of a circle is 2π times the radius

$$\text{torque} \times t = 2\pi \text{ (angular momentum).}$$

Therefore

$$t = \frac{2\pi\ (\text{angular momentum})}{\text{torque}}$$

$$= \frac{2\pi(2\pi r^2 sm)}{dmg}, \text{ substituting the expressions for angular momentum and torque}$$

$$= \frac{4\pi^2 r^2 s}{dg}, \text{ by cancelling m from top and bottom.}$$

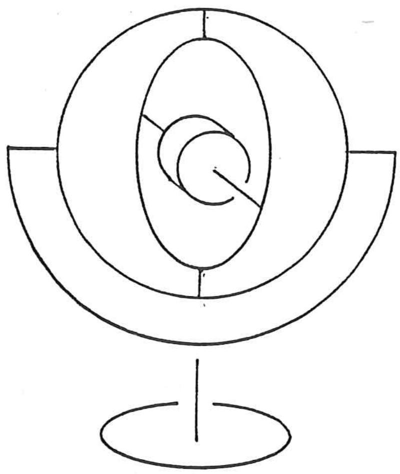
This completes the proof of the formula for t .

In Worksheet 1 Question 8 and in the experiments in the video Part 2 the values used were: r = 24.5 , g = 1000 .

experiment	1st	2nd	3rd
s	3	2	3
d	6.5	6.5	15
Hence t	11	7	5

Notice that the slower the spin the shorter the time because s is on the top of the formula, hence the faster the precession. Similarly the larger the distance the shorter the time and the faster the precession, because d is on the bottom of the formula.

If a gyro is freely suspended on gimbals as shown, then no torque can be applied to its axis however much the stand is moved around, and so there will be no precession. Therefore the spin axis will maintain a fixed direction in space, and can be used as the basis for a navigation system.

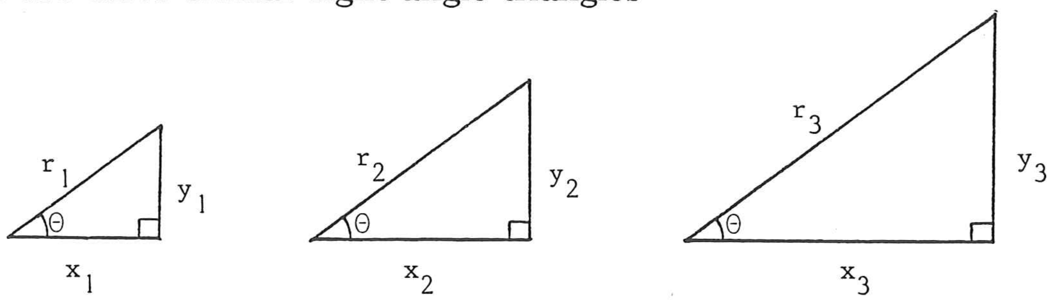


Gyroscopes and boomerangs

Worksheet 2

Do these questions after you have seen the video Part 2 and before watching Part 3. The first three questions are to introduce you to sin θ, in case you have not yet done any trigonometry. The next two questions are about spinning the wheel at an angle, to show that it does not make any difference to the precession time, as you saw at the end of the video Part 2. The last three questions are about a spinning top, and these are the most important questions to try before watching Part 3 of the video, because there you will see the top in slow motion doing what you have predicted. You will also see an animated version of the explanation, which will be much easier to follow if you have already thought about it beforehand.

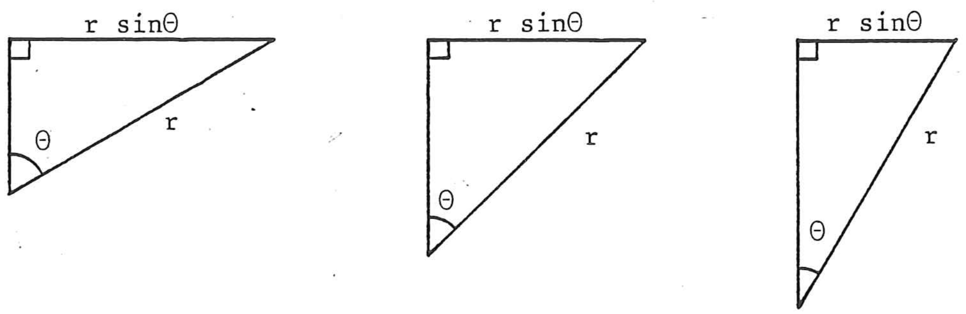
1. Here are three similar right-angle triangles



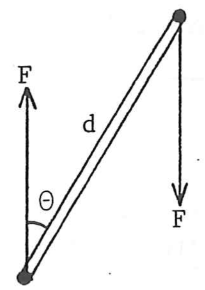
Measure θ , and use a calculator to find cos θ and sin θ . Measure the sides and verify that

$$\frac{x_1}{r_1} = \frac{x_2}{r_2} = \frac{x_3}{r_3} = \cos\theta, \qquad \frac{y_1}{r_1} = \frac{y_2}{r_2} = \frac{y_3}{r_3} = \sin\theta.$$

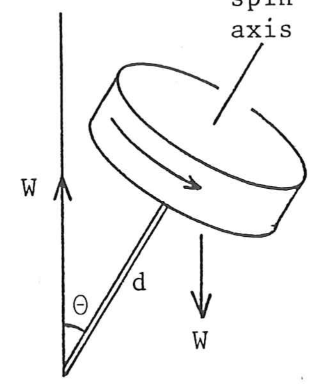
2. In each of the following right-angle triangles measure r and θ , calculate rsinθ , and verify that this is equal to the length of the side marked rsinθ .



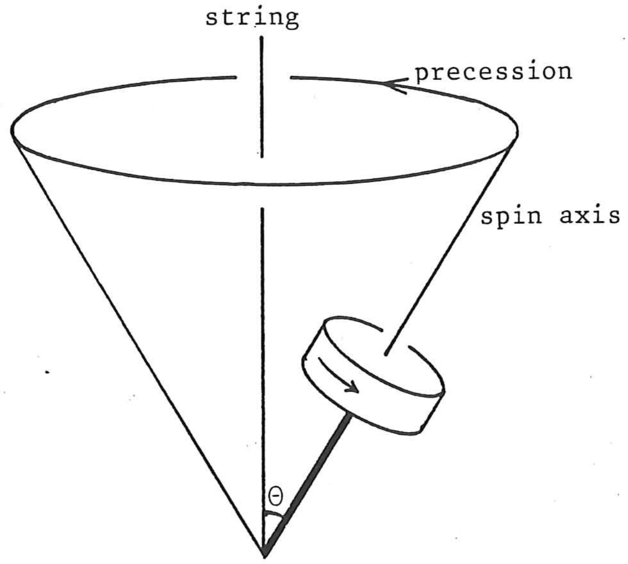
3. Two equal and opposite forces F are applied to the ends of a bar of length d at an angle θ. Write down the distance between the lines of action of the two forces, and the torque, in terms of d, θ and F. Measure d and θ, calculate the distance between the lines of action of the forces, and verify that you are correct by measuring it.



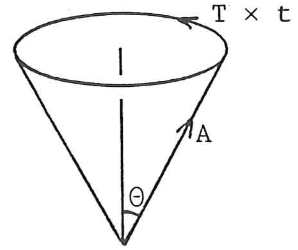
4. A spinning wheel is hanging from a string with its axis at an angle θ to the vertical.



Show that the torque $T = T_0 \sin\theta$, where T_0 denotes the torque when the spin axis is horizontal. What is the direction of the torque axis? Deduce that the spin axis precesses in a cone at a constant angle θ to the vertical.



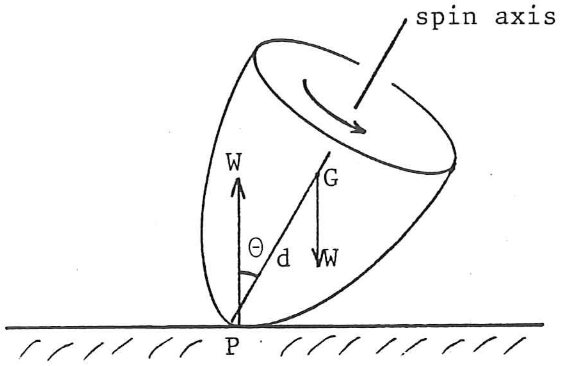
5. Let A denote the angular momentum in Question 4. Show that Newton's law gives a cone with side A and circumference $T \times t$, where t is the time to precess once round.



Show that $t = \dfrac{2\pi A \sin\theta}{T} = \dfrac{2\pi A}{T_0}$.

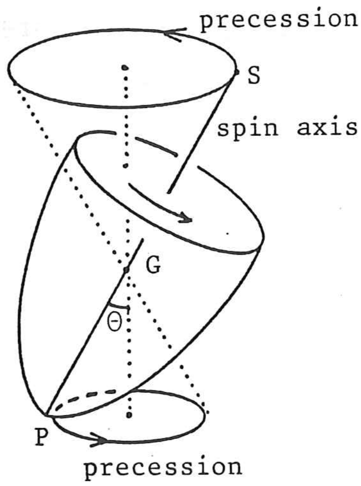
Deduce that the precession time t is independent of θ.

6. Suppose a top is spinning on a table with its axis inclined at an angle θ to the vertical as shown.

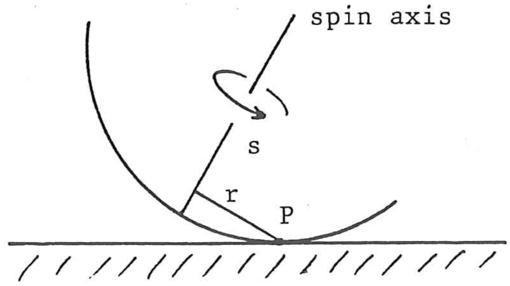


Ignoring friction, there are two forces acting on the top, the weight W downwards at the centre of gravity G and an equal and opposite upthrust at the point P in contact with the table. Since the forces are equal and opposite G remains stationary (for a proof see Appendix 1 Example 3). Let d be the distance from G to the tip. Show that the torque = Wdsinθ, approximately (only approximately because P is not quite at the tip).

What is the direction of the torque axis? Let S be a point on the spin axis. Show by the gyro law that S precesses in a horizontal circle with centre vertically above G.
Deduce that P precesses in a circle with centre vertically below G, and that the spin axis precesses in a double cone with vertex G, at constant angle θ to the vertical.



7. Now take friction into account as follows. The diagram shows an enlargement of the tip of the top with the point P a small distance from the spin axis.



Let r = distance of P from the spin axis
s = spin in revolutions per second
v_1 = speed of P moving away from you due to the spin.

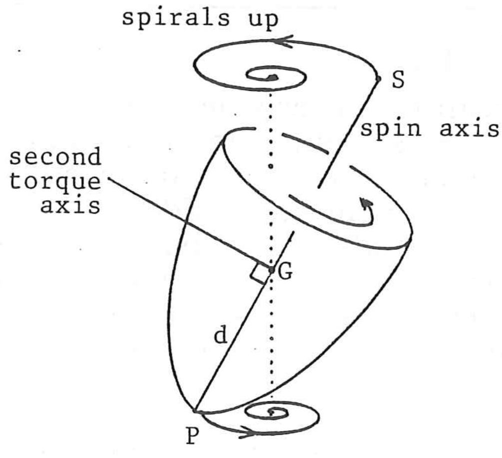
Show that $v_1 = 2\pi rs$.

Let R = radius of the precession circle of P
p = rate of precession of P round this circle in revolutions per second.
v_2 = speed of P moving towards you due to the precession.

Show that $v_2 = 2\pi Rp$.

Let $v = v_1 - v_2$
= net speed of P moving away from you.

Since s is much larger than p, but r is much smaller than R, it is not clear at first sight which of v_1 or v_2 will be the larger. Explain why the faster the spin the slower the precession. Deduce that if the top is spun sufficiently fast then $v > 0$. Therefore P is sliding on the table away from you, and friction causes an opposing force F towards you. The force F causes a second torque about G (in addition to the first torque described in Question 6). What is the size of this second torque about G ? [Hint: imagine using a spanner to tighten a nut at G, and pushing on the spanner with force F at P.]



Verify that the second torque axis is in the plane of the paper perpendicular to the spin axis (approximately), and in the upwards direction. Deduce by the gyro law that the spin axis will rise, at the same time as precessing, and hence will spiral towards the vertical. When the spin axis becomes vertical it is called a *sleeping* top.

[Remark: this is the only example in the book of a torque arising from a single force rather than from a pair of equal and opposite forces, and that is why we are careful to add the words "about G". For a proof of the gyro law in this case see Appendix 1 Theorem 3.]

8. What happens when the top slows down?

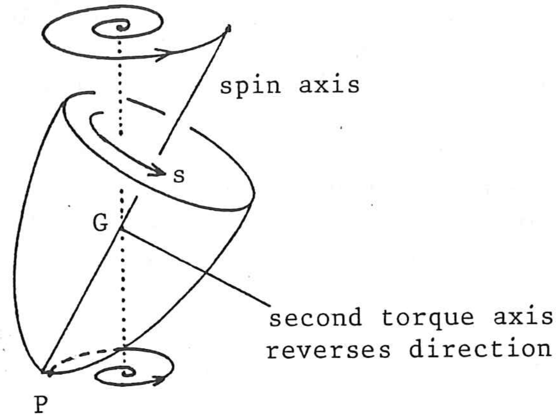
Gyroscopes and boomerangs

Notes on the Video Part 3: Tops, eggs and boomerangs

Tops going to sleep and waking up

People have played with tops ever since there have been flat surfaces on which to spin them. The sleeping top is particularly fascinating, and is often used as a metaphor for the paradox of stillness within motion. For example I once asked a Japanese Buddhist mathematician in Kyoto what he meant by "emptying his mind" because emptying the mind is common practice amongst meditative religions like Buddhism, and he replied that it was a "dynamic emptiness" like being at the still centre of a sleeping top, ready to resonate with faint echoes deep inside himself that were normally inaccessible to him.

A description of why a spinning top rises and goes to sleep was given in Worksheet 2 Questions 6 and 7. Here, continuing with the same notation as in those two questions, we answer Question 8 and describe what happens when the top slows down and wakes up.

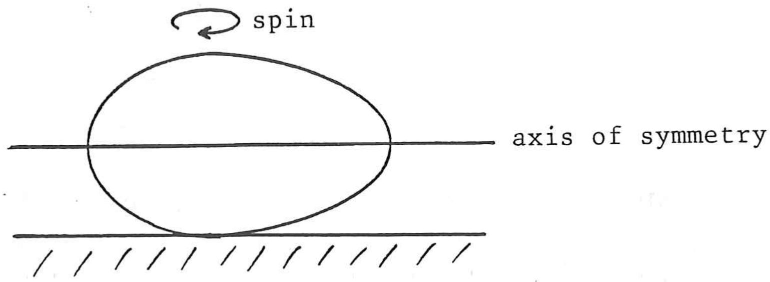


As the spin s decreases due to friction the angular momentum decreases, and so the precession time decreases (as can be seen from the formula for the precession time in Worksheet 2 Question 5). Therefore the rate of precession p increases. Eventually the speed v_1 of P away from you due to the spin falls below the speed v_2 towards you due to precession. Therefore the net speed $v = v_1 - v_2$ changes sign from positive to negative. Therefore the direction of sliding of P on the table is reversed, the friction F is reversed, and so the direction of the second torque axis is reversed, as shown in the diagram above. By the gyro law the spin axis will now chase it downwards instead of upwards. Therefore the spin axis suddenly begins to spiral outwards until it reaches an angle where $v_1 = v_2$ temporarily, and so $v = 0$. Therefore the top begins to roll instead of sliding, and to precess at a constant angle to the vertical. Since the rolling makes quite a loud noise the sleeping top is said to have "woken up". After a few moments the sliding begins again with $v<0$, and so then the top topples right over.

Spinning eggs standing on end

To get a spinning egg to stand on end it has to be hard-boiled, or made of some solid material like wood or marble so that it behaves as a rigid body. Otherwise if you try and spin an uncooked egg you will find that the liquid inside lags behind and causes the shell to slow down again almost at once, so that you can never get it to spin fast enough to rise up on end. The liquid inside will indeed be rotating, but only very slowly, and you can verify this by momentarily stopping the egg and immediately letting it go again: this will not stop the liquid inside, and so it will cause the shell to start rotating again slowly as soon as you let go.

If you spin a hardboiled egg fast enough then it will rise up and spin standing on end. At first sight this looks like a top going to sleep, but in fact the explanation turns out to be quite a bit harder. This is because when you spin an egg you don't spin it about its axis of symmetry as you do with wheels and tops, but rather you spin that axis of symmetry about the vertical.



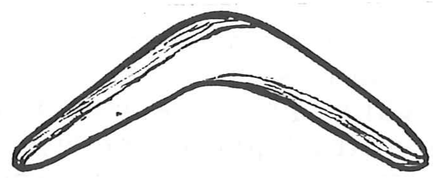
In other words the axis of symmetry has *fast* precession rather than the *slow* precession that we have been studying in wheels and tops. As a result the motion cannot be explained in terms of elementary mathematics like the other experiments in the video, but needs a more sophisticated treatment, which is given in Appendix 2.

Boomerangs

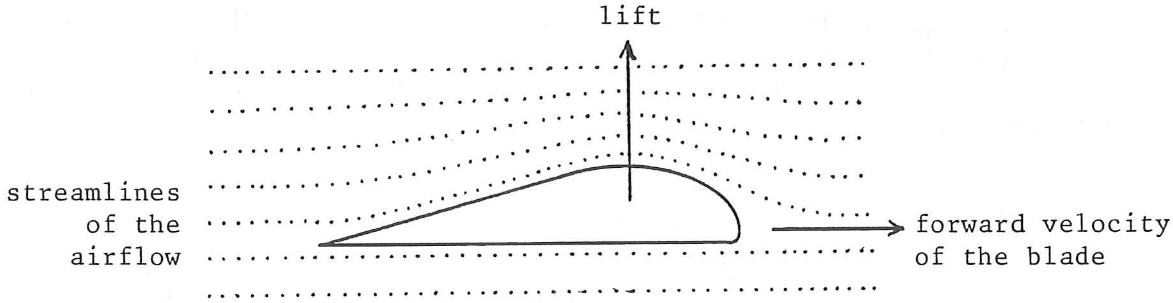
Boomerangs are even older than tops. The Australian Aborigines are the most famous world experts, and indeed the name boomerang comes from their language, but boomerangs were also invented independently in several other parts of the world. For example a 5000 year old boomerang was found in a bog in Denmark, a 7000 year old one was recently found in a cave in Poland, and when archaeologists opened Tutankhamun's tomb in Egypt they found amongst the treasures three ivory boomerangs with gold tips. The gold tips may have been functional as well as decorative by increasing the range of flight.

The great thing about a boomerang is that it flies in a circle and returns to you so that you can catch it. It achieves this by being both an aircraft and a gyro: being an aircraft enables it to fly, and being a gyro enables it to steer itself round the circle.

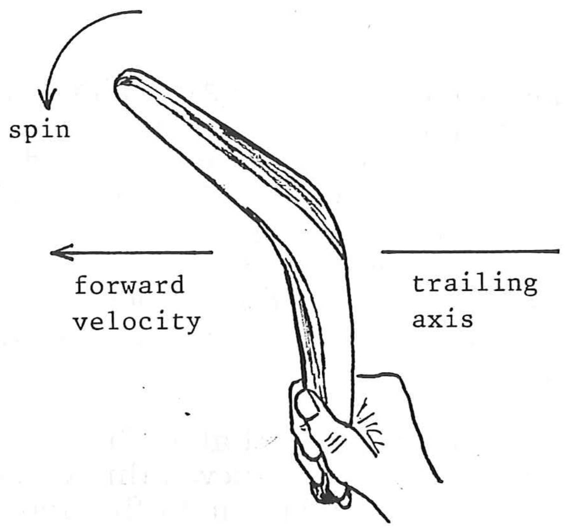
The traditionally shaped boomerang has two blades as shown, but boomerangs can also be made in large variety of other shapes including V, X, S, ✱. The one thrown in the video was cross-shaped with four blades, and instructions on how to make one like that are given in Worksheet 3 Question 5.

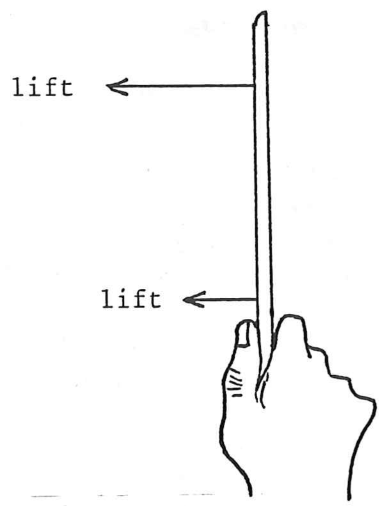


In cross-section each blade is shaped aerodynamically like an aeroplane wing, so that as it moves through the air it generates lift; in other words the air exerts a force on the blade perpendicular to the blade.



What actually causes the lift is the curvature of the upper surface of the blade, which induces the air to flow faster at a lower pressure over that surface; therefore there is less air pressure pushing down on the upper surface than there is pushing up on the lower surface, the difference giving the lift. In an aeroplane the wing is horizontal so that the lift is upwards, thus counteracting gravity and enabling the aeroplane to fly. A boomerang, however, is thrown forwards in a vertical plane and so the lift acts sideways.

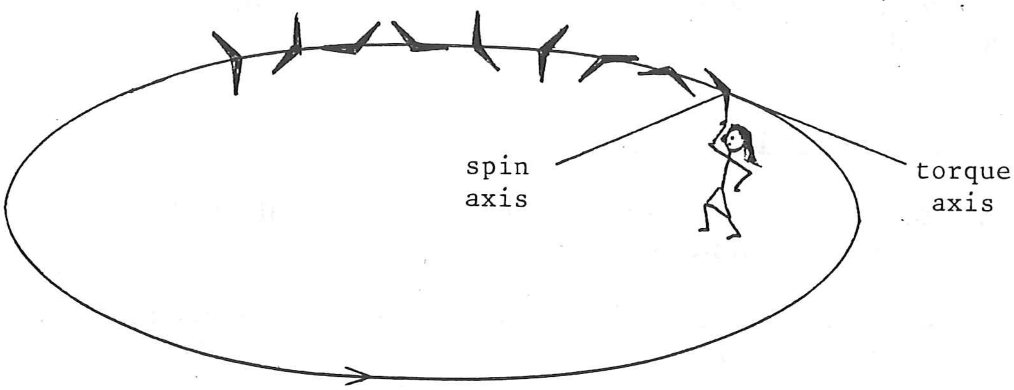




If you are right-handed hold the boomerang with the curved surface next to your face, so the lift will be to the left and the boomerang will fly in a circle to the left. A left-handed thrower needs a left-handed boomerang in which everything is reversed.

Any object moving steadily round a circle has an acceleration towards the centre of that circle and therefore requires a force towards the centre to produce that acceleration (see Worksheet 4 Question 3 and Appendix 1 Example 2). In the case of the boomerang this force is provided by the lift. In Worksheet 4 Question 4 the lift is calculated, and it is shown that the radius R of the flight circle is built into the boomerang, independent of the speed and the spin with which it is thrown. Let Ω_1 denote the rate of turning round the flight circle in radians per second, which is determined by the speed since R is fixed.

We next turn our attention to the gyroscopic effect. For a right-handed thrower the spin axis is horizontal and to the left. Now the upper blade is moving much faster through the air than the lower blade because for the upper blade the spin is *added* to the forward speed of the boomerang, whereas for the lower blade the spin is *subtracted* from the forward speed of the boomerang. Since the lift depends upon the square of the speed through the air the upper blade generates much more lift than the lower blade. Therefore the air exerts a torque on the boomerang as well as a force. By the right-hand screw rule the torque axis points backwards in the direction of the trailing axis (that is the opposite direction to the forward velocity). By the gyro law the spin axis will chase the torque axis and cause the boomerang to precess to the left.



Let Ω_2 denote the rate of precession in radians per second. The secret of the boomerang and the basic reason why it works is that

$$\Omega_1 = \Omega_2 \ .$$

In other words the boomerang steers itself around the flight circle at exactly the right rate, so that its plane is always tangent to the circle, and the lift is always pointing towards the centre of the circle. Otherwise if it were to precess too quickly or too slowly then it would rapidly stall and lose its ability to fly. The magic formula $\Omega_1 = \Omega_2$ is proved in Worksheet 4 Question 4.

So far we have ignored gravity. As the boomerang travels round its flight circle, it begins to fall downwards at the same time. Therefore the direction of the forward velocity begins to point slightly downwards, and so the trailing axis, and hence also the torque axis, begin to point slightly upwards. By the gyro law the spin axis chases it upwards and so the boomerang begins to "lay flat". By the time it returns to the thrower it should be spinning horizontally, and the forward speed will have dropped due to drag, and so you can catch it by clapping it between your flat hands.

The process of a boomerang laying flat is a bit like a top going to sleep, although this analogy is not very good because the underlying mathematics is quite different. A better analogy is as follows. The argument about the boomerang flying in a circle of radius R was independent of the original direction in which it was thrown, because gravity was not involved. Therefore, in whatever direction it is thrown, it will begin to fly on the surface of a huge imaginary sphere of radius R. Therefore it will behave like a marble rolling inside a hemispherical bowl. Just as the marble is subject to gravity and friction, and after following some path on the surface of the bowl will eventually settle down at the bottom of the bowl, so the boomerang will be subject to gravity and drag, and after following some path on its huge imaginary sphere should eventually settle down spinning horizontally at the bottom of its sphere, and then gently drift to earth.

You can experiment by projecting a marble on the inside surface of a bowl from various points and in various different directions, and watching the path that it then takes to the bottom. This will give you a prediction of how your boomerang will behave if you throw it at different angles above the horizontal, with its plane at different inclinations to the vertical.

Recommended further reading

1. M.J. Hanson, *The boomerang book*, Puffin Books (Penguin Books), Harmondsworth, Middlesex, 1974.

2. J. Jennings and N.H. Hardy, The boomerang and its flights, *World Wide Magazine*, London, 2 (1898) 626-629.

3. J. Perry, *Spinning Tops and gyroscopic motion*, 1916 (reprinted Dover Publications, 1957).

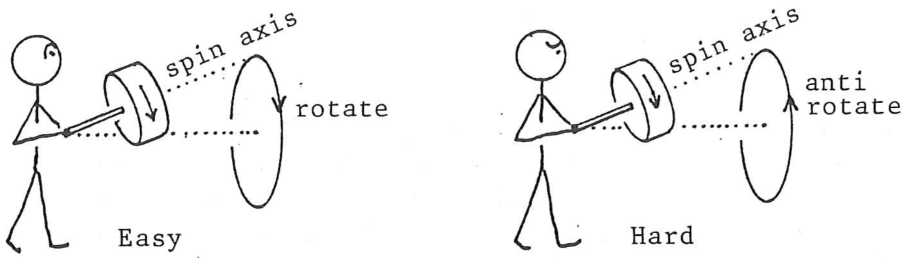
Gyroscopes and boomerangs

Worksheet 3

Worksheets 3 and 4 are for you to try after you have finished watching the video, to help reinforce some of the ideas. Worksheet 3 is the easier, and Worksheet 4 is the harder because some of the mathematics in it is a little more advanced.

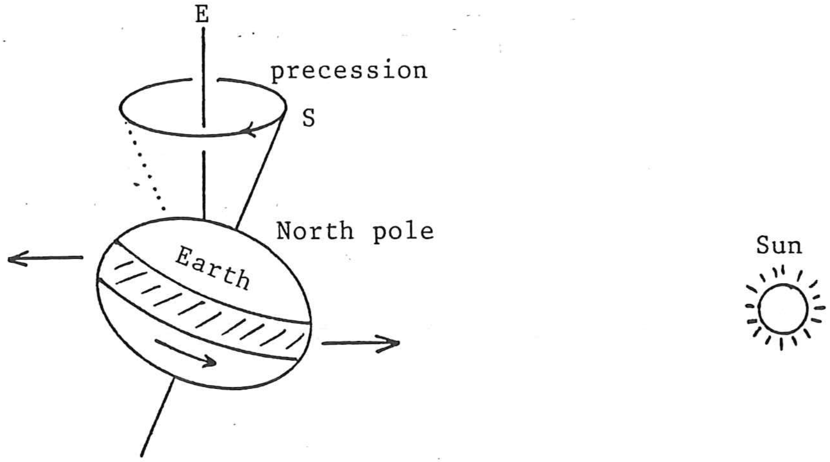
In Worksheet 3 the first question explains the first experiment in the video, the bizarre behaviour of a spinning wheel if you try and move its axis the wrong way. The next three questions are about astronomical gyros, the earth, the moon, and a spaceship. The last question is a practical exercise on how to make your own boomerang.

1. Imagine holding a spinning wheel and trying to rotate the spin axis in a cone as shown.



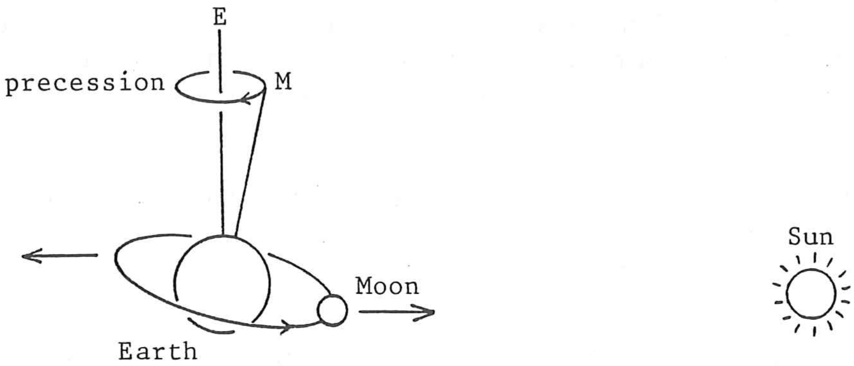
Explain why rotating it the same way as the spin axis is easy, but rotating it the opposite way is hard. Predict what will happen when you try and rotate it the opposite way. Then get a hold of a bicycle wheel and test your predictions.

2. Let S denote the spin axis of the earth, and let E be the axis perpendicular to the plane of the earth's orbit.



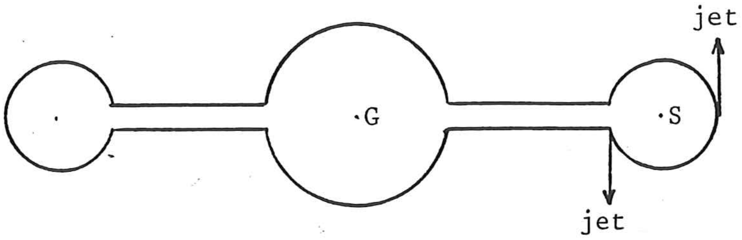
Centrifugal force makes the earth fatter at the equator than at the poles and so it is like a sphere with an extra band round the equator. The sun exerts a torque on this band because it attracts the nearer part more, and the further part less than the rest of the earth, as shown. Which direction is the torque axis? Deduce that S precesses westwards in a cone around E at a constant angle to E, as shown. [Note: the angle is 23.5° and it takes 45,000 years to go once round.]

3. Let M and E be axes perpendicular to the planes of the moon's orbit and the earth's orbit.



If we represent the average position of the moon by a ring spread evenly round its orbit then this ring acts as a gyro, as the earth did in the last question. Deduce that M precesses westwards in a cone around E at a constant angle to E. [Note: the angle is 5° and it takes 18.6 years to go once round. One can prove by calculating the angular momentum and average torque that if the angle is small then the precession time is 4/3 times the number of months in a year, approximately. This result is independent of the masses of the sun, earth and moon and the radii of their orbits, and so also works for the moons of Saturn & Jupiter.]

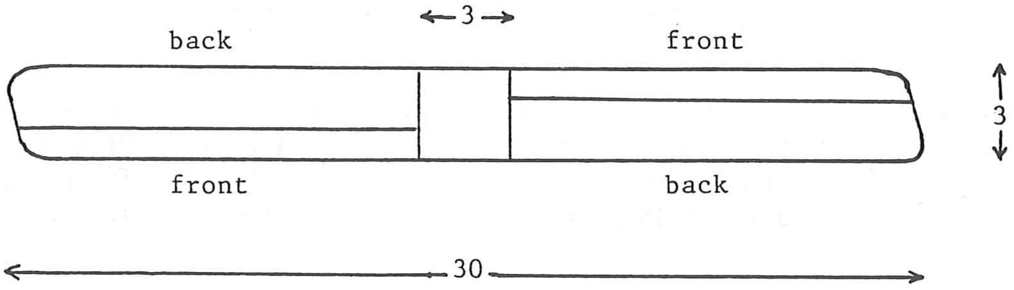
4. A space ship with centre of mass G and command satellite S is floating freely in space. It then squirts equal and opposite jets from the points shown, and in the directions shown, in order to start rotating.



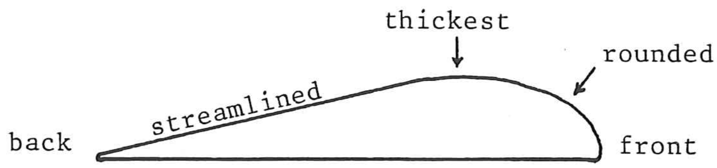
Which way does it begin to rotate, and about which point does it rotate?

5. **How to make a boomerang**

Get some balsa wood about 2 millimetres thick. Cut out two pieces, each about 30 cm long and 3 cm wide. Round the ends with sandpaper. Mark the top of each piece into two blades leaving 3 cm in the middle.

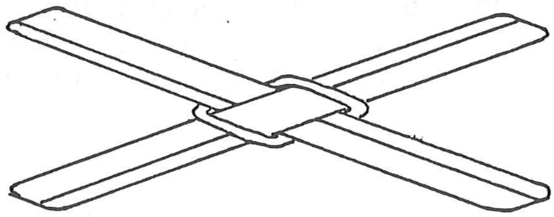


The front of each blade is identified by imagining rotating the piece anti-clockwise if you are right-handed, and clockwise if you are left-handed. The diagram is for a right-handed boomerang. Draw a line along each blade one third from the front. Sandpaper the top of each blade until it has cross-section like an aeroplane wing, rounded at the front, thickest where you drew the line, and streamlined at the back.



It will improve the flight if you can bend up the ends a little.

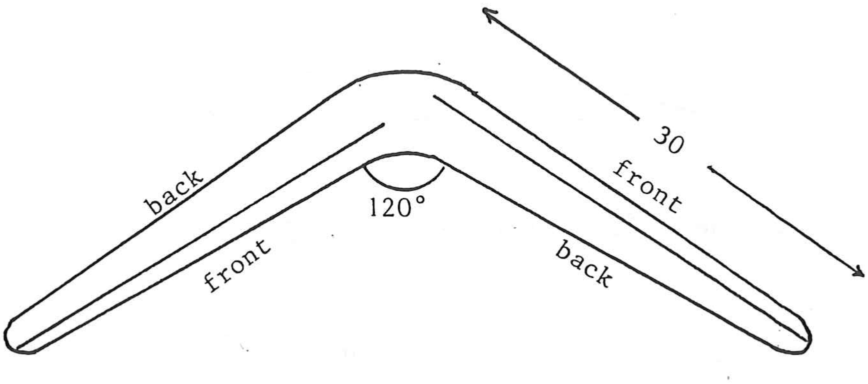
This can be done provided you are VERY CAREFUL NOT TO BURN YOURSELF by steaming the underside for a few seconds while holding it gently bent. I strongly recommend wearing gloves and getting an adult to help you. Finally put the two pieces together in the shape of a cross, and hold them together with an elastic band going over one piece and under the other.



Your boomerang is now ready to fly. When you throw it hold it vertically as described in Notes 3, and give it a good flick as it leaves your hand so that it starts spinning fast, and goes off in a direction about 10° above the horizontal. When it returns catch it by clapping it between your flat hands.

A balsa wood boomerang flies best in a large hall where there are no air currents and no breakable objects, and preferably no other people until you get skilled, although the balsa wood is so light that it is unlikely to hurt anyone if it hits them. If you throw a balsa wood boomerang out of doors the wind is liable to catch it and blow it all over the sky.

For outdoor throwing it is better to make a traditional shaped boomerang out of plywood because being heavier it will have a much larger flight circle and be less affected by the wind.



Use plywood about 6 mm thick. Each blade should be about 30 cm long and about 4-5 cm wide, and the angle between the blades about 120°. Use the same method of marking and shaping the blades, except that with plywood it is quicker to use a spokeshave to do most of the shaping and just use sandpaper for the final smooth finishing.

When throwing a plywood boomerang choose a still day and stand in the middle of a very large field with no other people around. Although a plywood boomerang only weighs about 100 grams it can nevertheless cause a serious injury if it hits you on return, so watch it like a hawk, and if anybody else is there insist that they watch it too. Throw it about 45° to the right of the direction from which the wind is blowing, and if you want to catch it wear gloves.

Gyroscopes and boomerangs

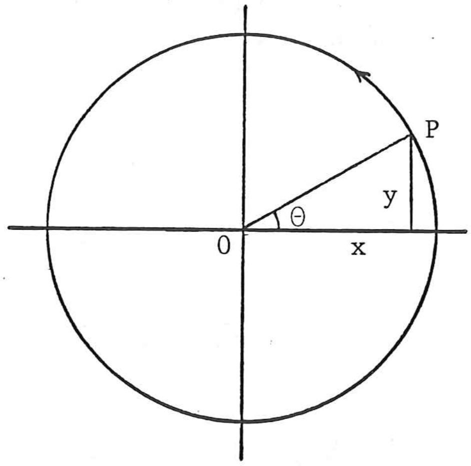
Worksheet 4

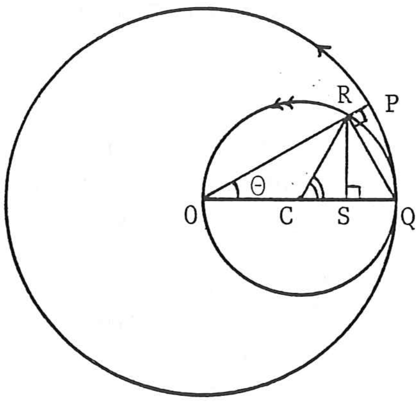
The purpose of Worksheet 4 is to prove the magic formula for boomerangs, $\Omega_1 = \Omega_2$. This will require slightly more advanced mathematics than the previous worksheets, and the first three questions are designed to help towards it. Question 1 calculates the average values of $\cos\theta$ and $\cos^2\theta$, and Question 2 verifies these results graphically. Question 3 calculates the acceleration of an object moving steadily round a circle. In the proof of the magic formula in Question 4 you need to calculate the lift, torque and angular momentum of a boomerang, which involves integrating along the blades. For this you need to use the two integrals

$$\int_0^b dr = b\ , \quad \int_0^b r^2\,dr = \frac{1}{3}b^3\ .$$

Apart from that the proof is just a long sequence of elementary steps.

1. The unit circle has centre the origin O and radius 1. Let P be a point on the unit circle with cartesian coordinates x,y and polar coordinate θ. Express x and y in terms of θ.





If P goes steadily round the unit circle then by symmetry the average value of x is zero. Deduce that the average value of $\cos\theta$ is zero.

Let Q be the position of P when $\theta = 0$.
Let R be the foot of the perpendicular from Q to OP.
Let S be the foot of the perpendicular from R to OQ.

Prove that

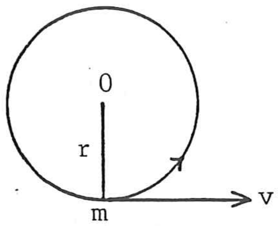
$$OS = \cos^2\theta.$$

Prove that R lies on the circle diameter OQ. Let C be the centre of this circle. Prove that the angle QCR = 2θ. Deduce that if P goes steadily round the unit circle then R will go steadily twice round the smaller circle. Deduce that the average value of $\cos^2\theta$ is 1/2. Find the average values of $\sin\theta$, $\sin^2\theta$, and $\cos\theta\sin\theta$.

2. Use a hand calculator to find $\cos\theta$, correct to two decimal places, for $\theta = 0^o, 15^o, 30^o, 45^o, 60^o, \ldots, 360^o$. Draw the graph of $\cos\theta$ by marking these points on graph paper and joining them up with a smooth curve. Verify that the average value of $\cos\theta$ is zero.

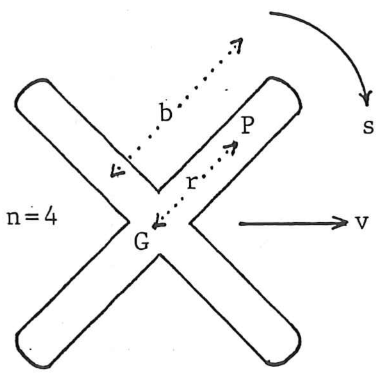
   Do the same for $\cos^2\theta$, and verify that its average value is 1/2.

3. Suppose a mass m is moving with speed v in a circle with centre O and radius r. Show that it has acceleration v^2/r towards O, and will require a force of mv^2/r towards O to ensure that it goes round the circle.



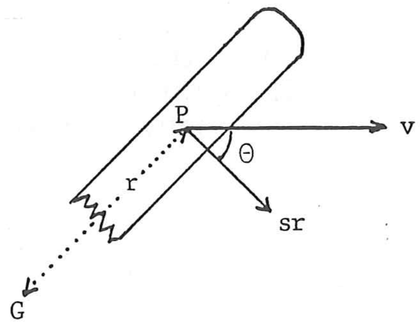
4. The purpose of this question is to prove the secret of the boomerang, that the rate of turning equals the rate of precession. Let

m = mass
G = centre of gravity
n = number of blades
b = length of each blade
P = a point on a blade
r = GP
v = forward speed
s = spin in radians/sec
λ = aerodynamic factor
L = total lift
T = total torque about G
A = angular momentum about G
R = radius of flight
Ω_1 = rate of turning round the flight circle in radians/sec
Ω_2 = rate of precession in radians/sec.

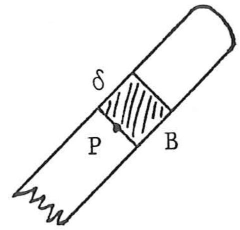


The speed of P relative to G is sr at right-angles to the blade. Let θ = angle between the directions of v and sr , as shown. Prove that

$$(\text{speed of P})^2 = v^2 + s^2r^2 + 2vsr\cos\theta.$$



Let B be the name of a small piece of the blade at P of length δ running from r to $r+\delta$. Since the total length of all the blades is nb , and since the aerodynamic factor is proportional to the length, deduce that



$$\text{aerodynamic factor of B} = \frac{\lambda\delta}{nb}.$$

The significance of the aerodynamic factor is that it is a constant, depending only upon the aerodynamic shape of the blade, that enables one to calculate the lift generated by that blade according to the formula:

$$\text{lift} = (\text{speed})^2 \times (\text{aerodynamic factor}).$$

The direction of the lift is perpendicular to the blade, and if the side of the boomerang facing you is flat while the other side is curved then the dirction of lift will be away from you. Deduce that B generates lift equal to

$$(v^2 + s^2r^2 + 2vsr\cos\theta)\frac{\lambda\delta}{nb}.$$

Use Question 1 to show that the average lift generated by B as the blade goes round is

$$(v^2 + s^2r^2)\frac{\lambda\delta}{nb}.$$

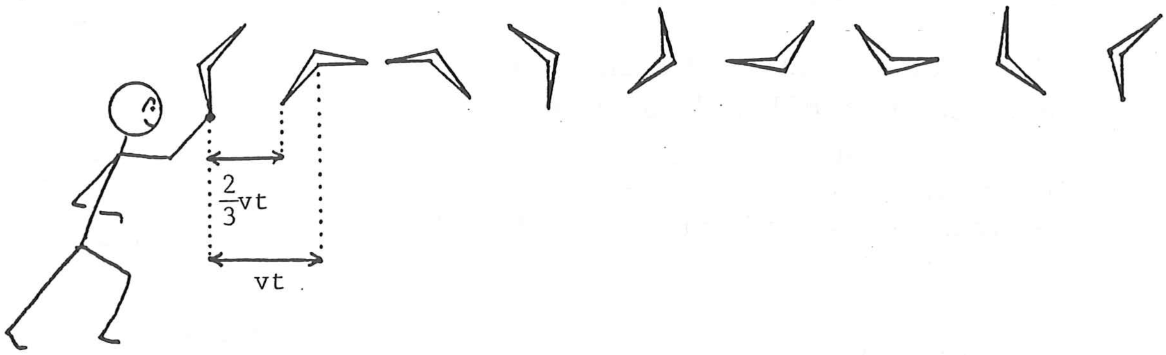
The total lift L is given by integrating along the blade and multiplying by the number of blades. More precisely replace δ by dr , and write the integral as

$$L = n\int_{r=0}^{b} (v^2 + s^2r^2)\frac{\lambda}{nb}dr.$$

Prove by evaluating the integral that

$$L = \lambda v^2\left[1 + \frac{1}{3}\left(\frac{sb}{v}\right)^2\right].$$

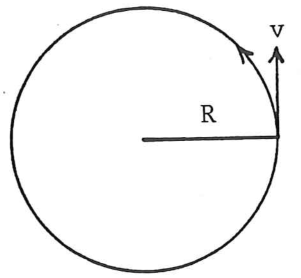
Assume that when the boomerang leaves the hand the forward speed of the end of the blade which was being held by the hand is 2v/3. Note: this can be shown experimentally by high speed time-lapse photography, as illustrated in the diagram, where t denotes the time lapse between two photos.



Calculate $\frac{1}{3}\left(\frac{sb}{v}\right)^2$, and deduce that initially $L = \lambda v^2$, approximately.

Since the boomerang is flying in a circle of radius R at speed v its acceleration towards the centre of the circle is v^2/R (see Question 3) and therefore by Newton's Law the lift required to produce that acceleration is

$$L = \frac{mv^2}{R} .$$



Deduce that

$$R = \frac{m}{\lambda} .$$

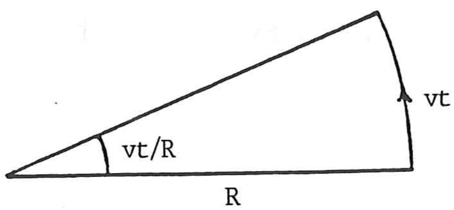
In other words the radius of flight is automatically built into the boomerang. The heavier it is the larger will be the flight circle, so to make a boomerang that will fly in a small circle inside a room use very light material like balsa wood. And the more aerodynamically efficient it is the larger λ will be, and the smaller will be the flight circle.

In time t the boomerang will fly a distance vt , and turn through an angle vt/R radians. Therefore its rate of turning is

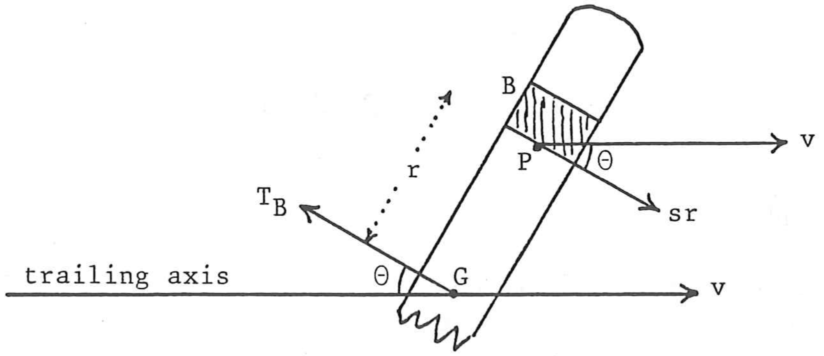
$$\Omega_1 = \frac{v}{R} \text{ radians per second.}$$

Deduce

$$\Omega_1 = \frac{v\lambda}{m} .$$



The next task is to compute the torque.



Show that the lift generated by B causes a torque T_B about G in the direction shown in the diagram, and given by

$$T_B = r\left(v^2 + s^2r^2 + 2vsr\cos\theta\right)\frac{\lambda\delta}{nb}.$$

The component of T_B in the direction of the trailing axis (that is in the opposite direction to v) is $T_B\cos\theta$. Deduce that this component is:

$$T_B\cos\theta = r\left(\left(v^2 + s^2r^2\right)\cos\theta + 2vsr\cos^2\theta\right)\frac{\lambda\delta}{nb}.$$

Use Question 1 to show that the average of this component as the blade goes round is:

$$\frac{vsr^2\lambda\delta}{nb}$$

Verify that the average component of T_B perpendicular to the trailing axis is zero. The total torque T about G is given by integrating along the blade and multiplying by the number of blades,

$$T = n\int_{r=0}^{b}\frac{vsr^2\lambda}{nb}dr .$$

in the direction of the trailing axis. Deduce that

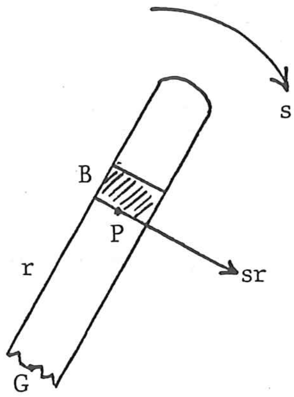
$$T = \frac{1}{3}vsb^2\lambda .$$

The next job is to calculate the angular momentum. Show that the mass of B is

$$\frac{m\delta}{nb}.$$

Deduce that the angular momentum of B about G is

$$r \times \frac{m\delta}{nb} \times sr$$

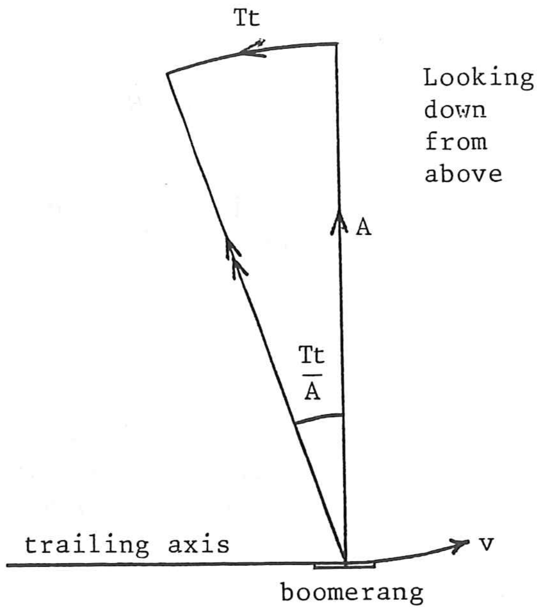


perpendicular to the paper in the direction away from you. The total angular momentum A about G is given by integrating along the blade and multiplying by the number of blades,

$$A = n\int_{r=0}^{b} \frac{msr^2}{nb}\,dr.$$

Deduce that

$$A = \frac{1}{3}msb^2.$$



By Newton's law of spinning motion the change in angular momentum in time t will be Tt; and so the boomerang will have precessed through an angle of Tt/A radians. Therefore the rate of precession Ω_2 is given by

$$\Omega_2 = \frac{T}{A} \text{ radians/sec.}$$

Deduce that

$$\Omega_2 = \frac{v\lambda}{m}.$$

Therefore $\Omega_1 = \Omega_2$, as required, and so the boomerang steers itself round the circle at exactly the right rate.

Note that this equation only holds for the first part of the flight while the forward speed v is greater than the rotational speed sb of the blade tips. During the flight the speed v will decrease due to the drag of the air on the blades. The spin s, however, will not decrease, because as long as $v > sb$ the lower blade will be going "backwards" through the air from the point of view of its aerodynamic shape, and therefore the drag on it will roughly balance the drag on the upper blade so that neither drag will affect the spin. That is why a boomerang maintains its spin. Eventually the speed v will become small, and so T, Ω_1 and Ω_2 will become small since they are proportional to v, but L will not become small because the other term that we previously ignored will become dominant:

$$L = \frac{1}{3}\lambda s^2 b^2 .$$

By this time the boomerang will be spinning horizontally, and the lift L will no longer be playing the role of providing the acceleration towards the centre of the flight circle, but instead will take on the role of counteracting gravity, enabling the boomerang to float gently downward rather than fall to earth.

Recommended further reading

F. Hess, The aerodynamics of boomerangs, *Scientific American* 219 (November 1968) 124-136.

Appendix 1

On Newton's Law of Motion

The aim of this appendix is to show that Newton's law for linear motion implies that for spinning motion. We shall introduce just enough calculus and vector products to do this rigorously. Do not worry if you have not done any calculus before because the introduction is self-contained and geometrical, giving both intuition and motivation. But first we need to verify that our version of Newton's law of linear motion is equivalent to the more familiar version involving acceleration.

Definition

Acceleration = rate of change of velocity.

Newton's Law

$$\text{Mass} \times \text{acceleration} = \text{force}. \qquad (*)$$

† Lemma 1

If the acceleration is constant then (*) is equivalent to the version of Newton's law of linear motion used in the video and Notes 1.

Proof

Since acceleration is constant it can be written:

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time}}.$$

$$\therefore \text{ acceleration} \times \text{time} = \text{change in velocity}$$

$$= \text{new velocity - old velocity.}$$

$$\therefore \text{ force} \times \text{time} = \text{mass} \times \text{acceleration} \times \text{time, by } (*)$$

$$= \text{mass} \times (\text{new velocity - old velocity})$$

$$= \text{new momentum - old momentum.}$$

$$\therefore \text{ new momentum} = (\text{old momentum}) + (\text{force} \times \text{time}),$$

which is the version used in the video and Notes 1.

---

† A "lemma" is a little theorem.

Notation

If the acceleration is not constant then to define it precisely we need to introduce calculus and take limits. That is why Newton invented calculus. We shall also switch into using symbols rather than words because they are more efficient, and hence easier to understand. Recall that a *vector* has both size and direction, whereas a *scalar* only has size. Throughout the two Appendices we shall use heavy type to denote vectors, and ordinary type to denote scalars. For example velocity $\mathbf{v}$ is a vector but mass m is a scalar. If $\mathbf{x}$ denotes a vector then x will denote its size (and not its direction). For example if $\mathbf{v}$ is the velocity of an object then v denotes its speed.

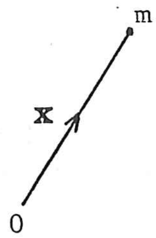
Definition

Suppose $\mathbf{x}$ is a vector that is varying with time, and suppose $\mathbf{x}$ changes to $\mathbf{x}'$ after time t. Define the *rate of change* $\dot{\mathbf{x}}$ of $\mathbf{x}$ by:

$$\dot{\mathbf{x}} = \text{limit } \frac{\mathbf{x}' - \mathbf{x}}{t} \qquad \text{as } t \to 0.$$

We also call $\dot{\mathbf{x}}$ the *derivative* of $\mathbf{x}$, and call the process of going from $\mathbf{x}$ to $\dot{\mathbf{x}}$ *differentiating with respect to* t.

Suppose now that $\mathbf{x}$ denotes the position of a mass m relative to a fixed origin O, in other words $\mathbf{x}$ is the vector Om. Then



$\mathbf{x}$ = position of m
$\dot{\mathbf{x}}$ = velocity of m = rate of change of position
$\ddot{\mathbf{x}}$ = acceleration of m = rate of change of velocity.

If m is subjected to a force $\mathbf{F}$ (which may also depend on time) we can write Newton's Law in symbols:

$$m\ddot{\mathbf{x}} = \mathbf{F}.$$

Example 1

The weight $\mathbf{W}$ of m is the force downwards on m due to the earth's gravitational attraction $\mathbf{g}$, and is proportional to m namely $\mathbf{W} = m\mathbf{g}$. By Newton's law the weight causes an acceleration $m\ddot{\mathbf{x}} = \mathbf{W}$. Therefore $\ddot{\mathbf{x}} = \mathbf{g}$, and so the acceleration downwards due to gravity is constant and the same for all objects.

Example 2

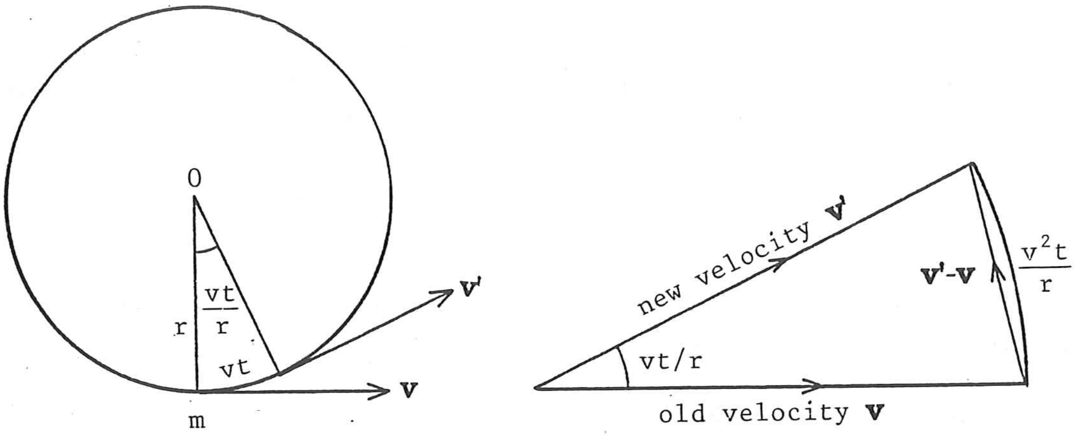
If a particle of mass m is moving with speed v in a circle of radius r and centre O then it has an acceleration v^2/r towards O, and will require a force of mv^2/r towards O to ensure that it goes round the circle.

Proof

By definition of derivative

$$\text{acceleration} = \text{limit}\ \frac{\mathbf{v'} - \mathbf{v}}{t} \quad \text{as } t \to 0.$$

Here $\mathbf{v}$ is the old velocity and $\mathbf{v'}$ the new velocity after time t, both having the same size v but in different directions. In time t the particle will have travelled round the circle a distance vt, and therefore the velocity will have changed direction through an angle of vt/r radians.



Therefore the vector $\mathbf{v'}$-$\mathbf{v}$ is approximated by a circular arc of length v^2t/r, and is approximately parallel to mO. Therefore

$$\frac{\mathbf{v'} - \mathbf{v}}{t} \text{ has } \begin{cases} \text{size approximately } v^2/r \\ \text{direction approximately towards O.} \end{cases}$$

As $t \to 0$ the size approaches v^2/r and the direction approaches the direction towards O. Therefore in the limit

$$\text{the acceleration has} \begin{cases} \text{size exactly equal to } v^2/r \\ \text{direction exactly towards O.} \end{cases}$$

By Newton's law,

$$\text{force} = \text{mass} \times \text{acceleration} = mv^2/r.$$

Lemma 2

If m is constant and $\mathbf{x}$ is a vector varying with time then

$$(m\mathbf{x})^{\cdot} = m\dot{\mathbf{x}}.$$

Proof

Suppose $\mathbf{x}$ changes to $\mathbf{x}'$ after time t.

$$\begin{aligned}\text{Then}\quad (m\mathbf{x})^{\cdot} &= \text{limit}\left[\frac{m\mathbf{x}' - m\mathbf{x}}{t}\right] \\ &= \text{limit}\ m\left[\frac{\mathbf{x}' - \mathbf{x}}{t}\right] \\ &= m\ \text{limit}\left[\frac{\mathbf{x}' - \mathbf{x}}{t}\right] \quad \text{, since m is constant,} \\ &= m\dot{\mathbf{x}}.\end{aligned}$$

Lemma 3

If $\mathbf{x},\mathbf{y}$ are two vectors varying with time then

$$(\mathbf{x} + \mathbf{y})^{\cdot} = \dot{\mathbf{x}} + \dot{\mathbf{y}}\ .$$

In other words the derivative of the sum is the sum of the derivatives, and this is also true for the sum of any number of vectors.

Proof

Suppose $\mathbf{x},\mathbf{y}$ change to $\mathbf{x}',\mathbf{y}'$ after time t. Then

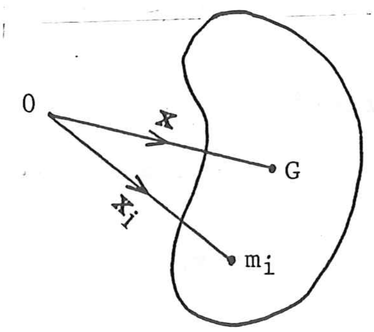
$$\begin{aligned}(\mathbf{x} + \mathbf{y})^{\cdot} &= \text{limit}\left[\frac{(\mathbf{x}' + \mathbf{y}') - (\mathbf{x} + \mathbf{y})}{t}\right] \\ &= \text{limit}\left[\frac{(\mathbf{x}' - \mathbf{x}) + (\mathbf{y}' - \mathbf{y})}{t}\right] \\ &= \text{limit}\left[\frac{\mathbf{x}' - \mathbf{x}}{t}\right] + \text{limit}\left[\frac{\mathbf{y}' - \mathbf{y}}{t}\right] \\ &= \dot{\mathbf{x}} + \dot{\mathbf{y}}\end{aligned}$$

Solid Bodies

Suppose that a body is made up of particles (or atoms) which are labelled $m_1, m_2, ..., m_n$. Here m_i denotes both the name and the mass of the i^{th} particle. The order in which they are labelled does not matter. The *total mass* m of the body is

$$m = \sum m_i$$

where $\sum$ denotes the sum over all i, $1 \le i \le n$. It is easiest to think of a rigid body, but it does not necessarily have to be rigid because it could be a swirling mass of liquid or a cloud of gas like the sun. Let $\mathbf{x}_i$ denote the position of m_i relative to a fixed origin O. The position $\mathbf{x}$ of the centre of gravity G is defined by the equation



$$m\mathbf{x} = \sum m_i \mathbf{x}_i \,.$$

Differentiating twice and using Lemmas 2 and 3 gives:

$$m\ddot{\mathbf{x}} = (m\mathbf{x})^{\cdot\cdot} = \left(\sum m_i \mathbf{x}_i\right)^{\cdot\cdot} = \sum m_i \ddot{\mathbf{x}}_i$$

Suppose that the body is now subjected to external forces $\mathbf{F}_i$ at m_i. For example the forces may include gravity acting on each m_i as well as other external forces acting at particular points of the body. Define the *total force* acting on the body to be

$$\mathbf{F} = \sum \mathbf{F}_i.$$

It may seem a peculiar thing to do at first sight to add together all the forces, ignoring the points where each is acting, but the definition is justified by the following theorem.

Theorem 1

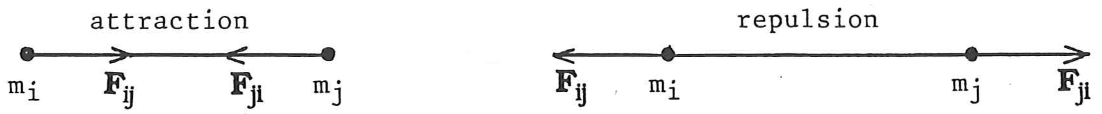
$m\ddot{\mathbf{x}} = \mathbf{F}$. In other words the centre of gravity G moves as if all the forces were acting together at G.

Proof

As well as the external forces there will also be internal forces between the particles keeping the body together. Let $\mathbf{F}_{ij}$ denote the force exerted on m_i by m_j, where $i \neq j$. We shall assume that $\mathbf{F}_{ij}$ acts along the line $m_i m_j$, either towards m_j (attraction) or away from m_j (repulsion). We also assume that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

In other words each pair of particles either attract each other with equal and opposite forces, or repel each other with equal and opposite forces. Newton summarised it as "action and reaction are equal and opposite".



Apply Newton's law to m_i:

$$m_i\ddot{\mathbf{x}}_i = \mathbf{F}_i + \sum_j \mathbf{F}_{ij} .$$

Sum over i

$$\sum_i m_i\ddot{\mathbf{x}}_i = \sum_i \mathbf{F}_i + \sum_{ij} \mathbf{F}_{ij} .$$

Now $\sum_i m_i\ddot{\mathbf{x}}_i = m\ddot{\mathbf{x}}$, as shown above.

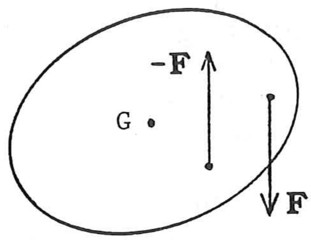
$\sum_i \mathbf{F}_i = \mathbf{F}$, the total force.

$\sum_{ij} \mathbf{F}_{ij} = \mathbf{0}$, because $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$ for each pair $i \neq j$.

Therefore $m\ddot{\mathbf{x}} = \mathbf{F}$, as required.

Example 3

If a body is subjected to equal and opposite forces then the total force will be zero (although the torque will not be zero if the forces are not in the same line). Therefore the centre of gravity G will have zero acceleration. Therefore if G is still initially then it will remain still.

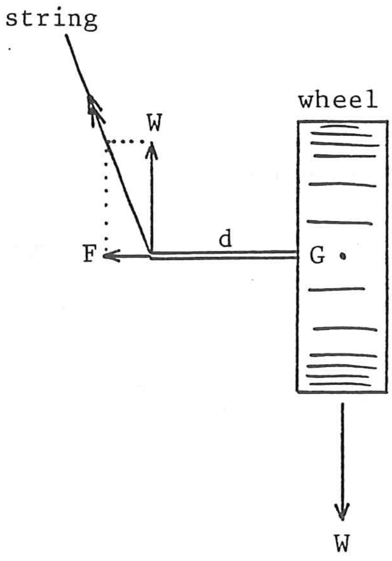


Example 4

If a spinning bicycle wheel is hung from a string as in the video then it will precess slowly round, and G will move slowly round a circle. Therefore by Example 2 and Theorem 1 there must be a small total force towards the centre of the circle. Therefore our pictures and calculations in Notes 1 and 2 and Worksheet 1 are not strictly accurate, because in those we mentioned only the torque. However, we shall now calculate the correction and show it is so small that it can be ignored.

What actually happens is that the string hangs at a slight angle, as is shown exaggerated in the diagram on the next page.

The pull of the string is the sum of a force W upwards equal and opposite to the weight of the wheel, and a small horizontal force F = mv^2/d, where m is the mass of the wheel, v the speed of G, and d the distance of G from the string. Using the notation and measurements of Notes 2,



$$v = \frac{2\pi d}{t} = \frac{d^2 g}{2\pi r^2 s}$$

$$\therefore \quad F = \frac{mv^2}{d} = \frac{md^3 g^2}{4\pi^2 r^4 s^2}$$

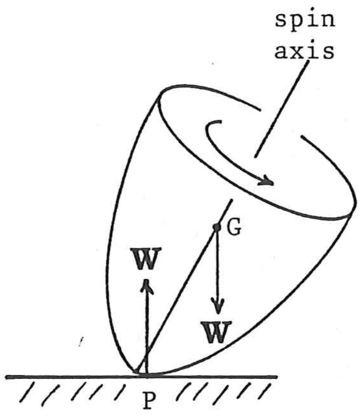
$$\therefore \quad \frac{F}{W} = \frac{d^3 g}{4\pi^2 r^4 s^2} = 0.002.$$

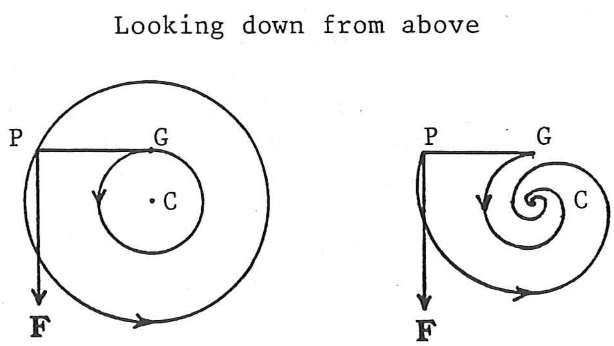
Therefore the angle of the string to the vertical is only one tenth of a degree, small enough to be ignored.

Example 5

In a spinning top without friction there are only two forces, the weight (which by Theorem 1 is equivalent to a single force **W** at the centre of gravity G) and an equal and opposite upthrust of the table at the point of contact P, forming a torque. Therefore the total force is zero. Therefore G is still, and the spin axis precesses in a cone around the vertical through G, keeping a constant angle to the vertical.

If, however, there is friction at P then the frictional force **F** acts towards you in the left diagram.





Therefore the total force = **F**. Therefore, by Example 2 and Theorem 1, if the spin axis were to precess at a constant angle to the vertical then G would move in a small horizontal circle centre C, where GC is parallel to **F**, as shown in the middle diagram. However, **F** also exerts a torque about G, as explained in Notes 2 and Theorem 3 below, which causes the spin axis to rise and the top to go to sleep. Therefore G performs a gently rising spiral towards its sleeping position C, as shown in the right diagram.

Example 6

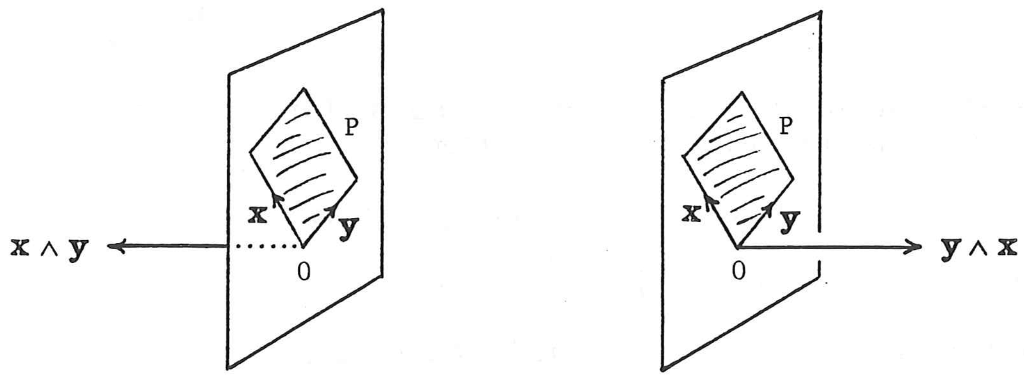
Suppose a boomerang of mass m is flying in a circle of radius R at speed v. If the boomerang has aerodynamic factor λ then, provided v is large enough, the total lift will be approximately λv^2 towards the centre of the circle. Therefore by Example 2 and Theorem 1,

$$\lambda v^2 = \frac{mv^2}{R}$$

Therefore $R = m/\lambda$, and so the radius of flight is built into the boomerang, independent of the speed v with which it is thrown (provided v is large enough).

Definition of vector product

To describe the spinning motion of solid bodies in general we shall need vector products, which are defined as follows. Given two vectors **x**, **y** let P be a parallelogram with sides **x** and **y**. Define the *vector product* $\mathbf{x} \wedge \mathbf{y}$ to be the vector of size equal to the area of P and in the direction perpendicular to the plane containing P given by the right-hand screw rule going from **x** to **y** by the shorter route.



Some books use the ordinary multiplication symbol $\mathbf{x} \times \mathbf{y}$ to indicate the vector product, but I prefer the notation $\mathbf{x} \wedge \mathbf{y}$ to emphasise that it is something rather special, particularly if this is the first time you have met it. If you want to read it (either out loud or quietly to yourself inside your head) you can say "**x** cross **y**".

Theorem 2

The main properties of the vector product are:

(i) $\mathbf{x} \wedge \mathbf{y} = -(\mathbf{y} \wedge \mathbf{x})$

(ii) $(m\mathbf{x}) \wedge \mathbf{y} = \mathbf{x} \wedge (m\mathbf{y}) = m(\mathbf{x} \wedge \mathbf{y})$, for any constant m.

(iii) If $\mathbf{x}$ is parallel to $\mathbf{y}$ or $-\mathbf{y}$ then $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$.
In particular $\mathbf{x} \wedge \mathbf{x} = \mathbf{0}$.

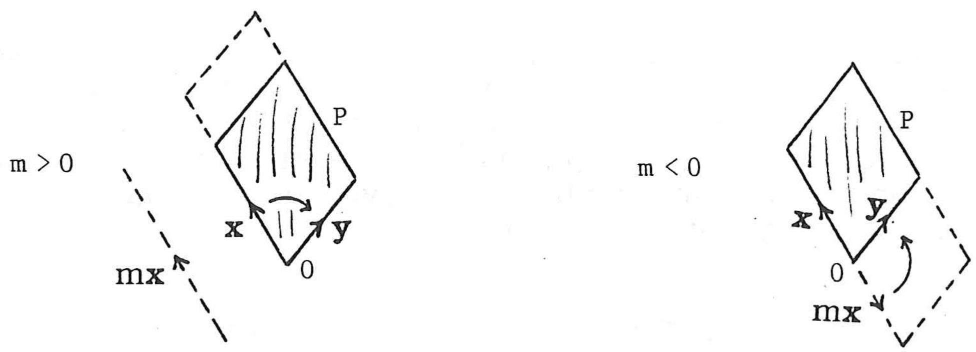
(iv) $\mathbf{x} \wedge (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) + (\mathbf{x} \wedge \mathbf{z})$
$(\mathbf{x} + \mathbf{y}) \wedge \mathbf{z} = (\mathbf{x} \wedge \mathbf{z}) + (\mathbf{y} \wedge \mathbf{z})$

(v) $(\mathbf{x} \wedge \mathbf{y})^{\cdot} = (\dot{\mathbf{x}} \wedge \mathbf{y}) + (\mathbf{x} \wedge \dot{\mathbf{y}})$

Proof

(i) Screwing from $\mathbf{y}$ to $\mathbf{x}$ is the reverse of screwing from $\mathbf{x}$ to $\mathbf{y}$, and therefore the direction is reversed, as in the diagram above. The size is the same because P is unchanged. Therefore $\mathbf{x} \wedge \mathbf{y} = -(\mathbf{y} \wedge \mathbf{x})$.

(ii) If $m > 0$ then multiplying $\mathbf{x}$ by m causes the area of P to be multiplied by m, but does not reverse the direction. Therefore $(m\mathbf{x}) \wedge \mathbf{y} = m(\mathbf{x} \wedge \mathbf{y})$.



If $m < 0$ then the area of P is multiplied by $(-m)$, and the direction of screw from $m\mathbf{x}$ to $\mathbf{y}$ is reversed. Therefore

$$(m\mathbf{x}) \wedge \mathbf{y} = -(-m)(\mathbf{x} \wedge \mathbf{y}) = m(\mathbf{x} \wedge \mathbf{y}).$$

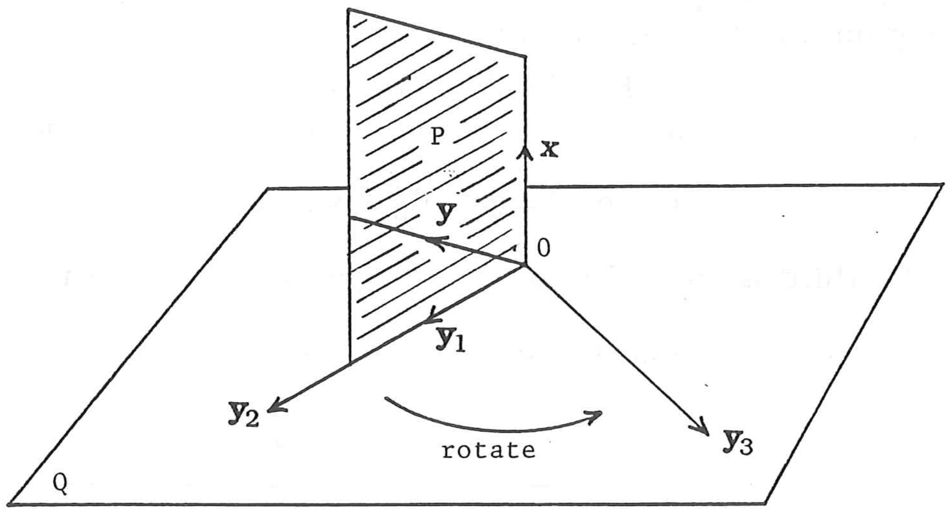
Similarly $(\mathbf{x} \wedge m\mathbf{y}) = m(\mathbf{x} \wedge \mathbf{y})$.

(iii) If $\mathbf{x}$ is parallel to $\mathbf{y}$ or $-\mathbf{y}$ then P collapses to a line with zero area. Therefore $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$.

(iv) We shall show that the map $\mathbf{y} \to \mathbf{x} \wedge \mathbf{y}$ is the composition of three maps

$$\mathbf{y} \to \mathbf{y_1} \to \mathbf{y_2} \to \mathbf{y_3}$$

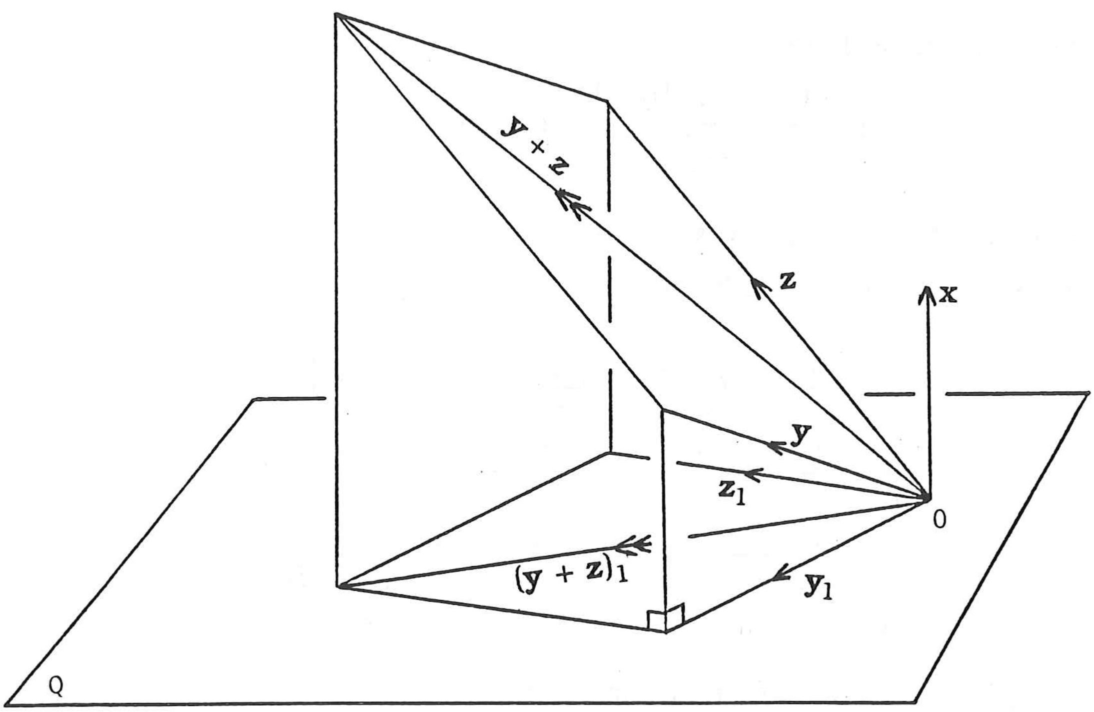
as follows. Let Q denote the plane through O perpendicular to $\mathbf{x}$. Let $\mathbf{y_1}$ be the projection of $\mathbf{y}$ onto Q.



Let $\mathbf{y_2} = x\mathbf{y_1}$, where x denotes the size of $\mathbf{x}$. Finally to obtain $\mathbf{y_3}$ rotate Q about O through 90^o in the direction of a right-hand screw about $\mathbf{x}$. We shall show that $\mathbf{y_3} = \mathbf{x} \wedge \mathbf{y}$, as follows. The parallelogram P can be regarded having base of length x and height y_1. Therefore area P = $xy_1 = y_2 = y_3$.

By construction $\mathbf{y_3}$ is perpendicular to the plane containing $\mathbf{x}$, $\mathbf{y_1}$, $\mathbf{y}$ and hence P. Finally $\mathbf{y_3}$ is in the direction of a right-hand screw going from $\mathbf{x}$ to $\mathbf{y}$ by the shorter route. Hence $\mathbf{y_3} = \mathbf{x} \wedge \mathbf{y}$, as desired.

Now each of these maps sends parallelograms to parallelograms, and hence preserves vector sums, as follows.



Firstly the parallelogram with sides $\mathbf{y}$, $\mathbf{z}$ is projected onto a second parallelogram with sides $\mathbf{y}_1$, $\mathbf{z}_1$ and

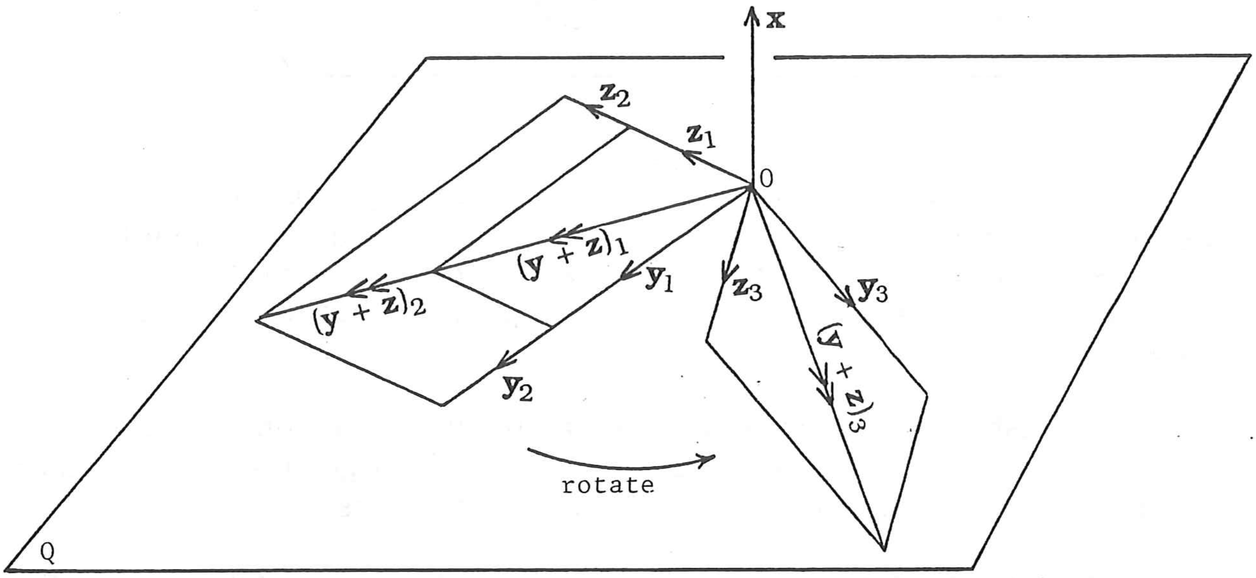
$$\text{diagonal} = (\mathbf{y} + \mathbf{z})_1 = \mathbf{y}_1 + \mathbf{z}_1.$$

The second is expanded by x into a third parallelogram with sides $\mathbf{y}_2$, $\mathbf{z}_2$ and

$$\text{diagonal} = (\mathbf{y} + \mathbf{z})_2 = \mathbf{y}_2 + \mathbf{z}_2.$$

Finally the third is rotated into a fourth parallelogram with sides $\mathbf{y}_3$, $\mathbf{z}_3$ and

$$\text{diagonal} = (\mathbf{y} + \mathbf{z})_3 = \mathbf{y}_3 + \mathbf{z}_3.$$



In other words $\mathbf{x} \wedge (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) + (\mathbf{x} \wedge \mathbf{z})$, as required.

To prove the second part:

$$\begin{aligned}(\mathbf{x} + \mathbf{y}) \wedge \mathbf{z} &= -\mathbf{z} \wedge (\mathbf{x} + \mathbf{y}), \quad \text{by (i)},\\ &= -(\mathbf{z} \wedge \mathbf{x}) - (\mathbf{z} \wedge \mathbf{y}), \quad \text{by the first part},\\ &= (\mathbf{x} \wedge \mathbf{z}) + (\mathbf{y} \wedge \mathbf{z}), \quad \text{by (i)}.\end{aligned}$$

(v) Suppose $\mathbf{x}$, $\mathbf{y}$ change to $\mathbf{x}'$, $\mathbf{y}'$ after time t. Then

$$\begin{aligned}(\mathbf{x} \wedge \mathbf{y})^{\cdot} &= \text{limit}\left[\frac{(\mathbf{x}' \wedge \mathbf{y}') - (\mathbf{x} \wedge \mathbf{y})}{t}\right], \text{ as } t \to 0,\\ &= \text{limit}\left[\frac{(\mathbf{x}' \wedge \mathbf{y}' - \mathbf{x} \wedge \mathbf{y}') + (\mathbf{x} \wedge \mathbf{y}' - \mathbf{x} \wedge \mathbf{y})}{t}\right]\\ &= \text{limit}\left[\frac{((\mathbf{x}' - \mathbf{x}) \wedge \mathbf{y}') + (\mathbf{x} \wedge (\mathbf{y}' - \mathbf{y}))}{t}\right], \quad \text{by (iv)},\\ &= \text{limit}\left[\left(\frac{\mathbf{x}' - \mathbf{x}}{t}\right) \wedge \mathbf{y}'\right] + \text{limit}\left[\mathbf{x} \wedge \left(\frac{\mathbf{y}' - \mathbf{y}}{t}\right)\right]\\ &= (\dot{\mathbf{x}} \wedge \mathbf{y}) + (\mathbf{x} \wedge \dot{\mathbf{y}}), \quad \text{since limit } \mathbf{y}' = \mathbf{y}.\end{aligned}$$

This completes the proof of Theorem 2.

We shall now use the vector product to define angular momentum and torque, and to prove Theorem 3 which is Newton's law of spinning motion for solid bodies. In order that the result be applicable to the examples in the video of the wheel, top and boomerang we shall need to take the origin at the centre of gravity G, even though G itself may be moving. To emphasise this point we shall use the notation $\mathbf{y}_i$ rather than $\mathbf{x}_i$ for the vector describing the position of the particle m_i relative to G. The only place where it will matter that the origin is moving is when we have to apply Newton's law to the individual particle m_i, and here we shall take into account any acceleration that G may have and show that *because* G is the centre of gravity it does not affect the result.

Angular momentum and torque for a particle

Given a particle m and a point G, let $\mathbf{y}$ denote the position of m relative to G (in other words the vector going from G to m). Then $\dot{\mathbf{y}}$ is the velocity of m relative to G. Define the

$$\textit{angular momentum of m about } G = \mathbf{y} \wedge m\dot{\mathbf{y}}.$$

If a force $\mathbf{F}$ acts on m, define

$$\textit{torque about } G = \mathbf{y} \wedge \mathbf{F}.$$

Compare these rigorous definitions in terms of vectors with the more elementary definitions in terms of words that were used to first introduce the concepts in the video:

$$\text{angular momentum} = \text{distance} \times \text{momentum}$$

$$\text{torque} = \text{distance} \times \text{force}.$$

Angular momentum of a solid body

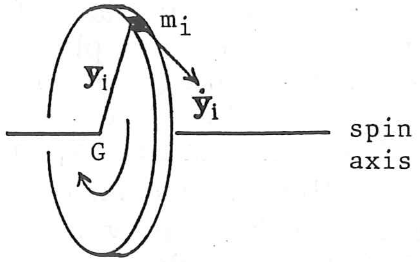
Suppose that a solid body consists of particles m_i, $1 \le i \le n$, and let G denote the centre of gravity. Let $\mathbf{y}_i$ denote the position of m_i relative to G. Then $\Sigma m_i \mathbf{y}_i = \mathbf{0}$ because G is the centre of gravity. Define the angular momentum $\mathbf{A}$ of the body about G by

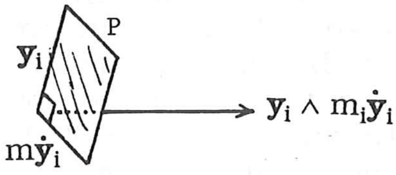
$$\mathbf{A} = \sum_i \mathbf{y}_i \wedge m_i \dot{\mathbf{y}}_i.$$

The following example shows that in the special case of a spinning bicycle wheel $\mathbf{A}$ agrees with the definition used in the video and Notes 2.

Example 7

A wheel of radius r and mass m, with the mass concentrated at the rim, is spinning at s revs/sec.





Each particle m_i at $\mathbf{y}_i$ on the rim will have velocity $\dot{\mathbf{y}}_i$ at right-angles to $\mathbf{y}_i$, and speed $2\pi rs$. Therefore

$$\mathbf{y}_i \wedge m_i\dot{\mathbf{y}}_i \text{ has } \begin{cases} \text{size } r \times 2\pi r s m_i \\ \text{direction parallel to the spin axis} \end{cases}$$

Summing over i:

$$\mathbf{A} = \sum \mathbf{y}_i \wedge m_i\dot{\mathbf{y}}_i = 2\pi r^2 sm \text{ along the spin axis.}$$

The torque on a solid body

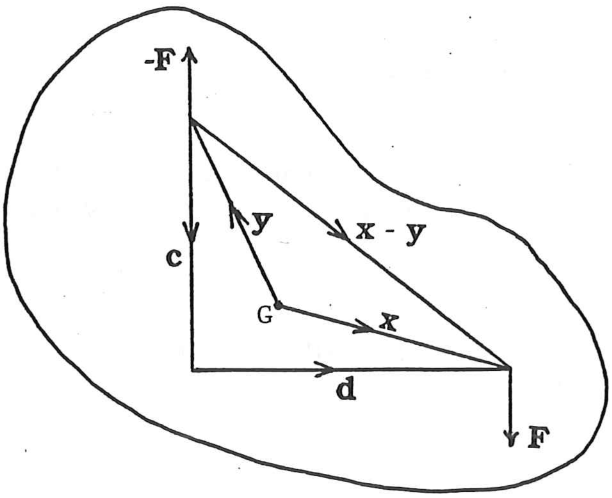
If external forces $\mathbf{F}_i$ act at $\mathbf{y}_i$ define the *total torque* $\mathbf{T}$ *about* G by:

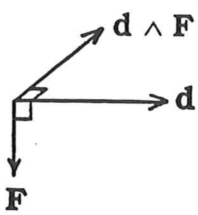
$$\mathbf{T} = \sum \mathbf{y}_i \wedge \mathbf{F}_i.$$

The following example shows that in the special case of two equal and opposite forces $\mathbf{T}$ agrees with the definition used in the video and Notes 1.

Example 8

Suppose a body is subjected to forces $\mathbf{F}$ at $\mathbf{x}$ and $-\mathbf{F}$ at $\mathbf{y}$.
Write $\mathbf{x} - \mathbf{y} = \mathbf{c} + \mathbf{d}$, where $\mathbf{c}$ is parallel to $\mathbf{F}$ and $\mathbf{d}$ at right-angles to $\mathbf{F}$.





Then the total torque **T** about G is given by:

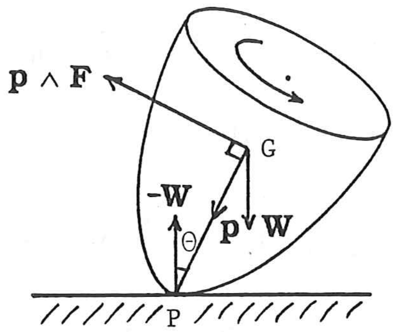
$$
\begin{aligned}
\mathbf{T} &= (\mathbf{x} \wedge \mathbf{F}) + (\mathbf{y} \wedge (-\mathbf{F})) \\
&= (\mathbf{x} \wedge \mathbf{F}) - (\mathbf{y} \wedge \mathbf{F}), \text{ by Theorem 2 (ii)} \\
&= (\mathbf{x} - \mathbf{y}) \wedge \mathbf{F}, \text{ by Theorem 2 (iv)} \\
&= (\mathbf{c} + \mathbf{d}) \wedge \mathbf{F}, \text{ since } \mathbf{x} - \mathbf{y} = \mathbf{c} + \mathbf{d} \\
&= (\mathbf{c} \wedge \mathbf{F}) + (\mathbf{d} \wedge \mathbf{F}), \text{ by Theorem 2 (iv)} \\
&= \mathbf{d} \wedge \mathbf{F}, \text{ because } \mathbf{c} \wedge \mathbf{F} = \mathbf{0} \text{ by Theorem 2 (iii)}
\end{aligned}
$$

Therefore T = dF, since **d** is at right-angles to **F**; and **T** is perpendicular to the plane containing the two forces in the direction given by the right-hand screw rule. Hence **T** agrees with the definition of torque given in the video.

Example 9

In a spinning top there are three forces acting:

(i) the weight **W** at G,
(ii) the upthrust -**W** at P, the point of contact,
(iii) the friction **F** at P, towards you.



Let **p** denote the vector GP, and θ the angle between **p** and the vertical. The total torque **T** about G is given by

$$
\begin{aligned}
\mathbf{T} &= (\mathbf{O} \wedge \mathbf{W}) + (\mathbf{p} \wedge (-\mathbf{W})) + (\mathbf{p} \wedge \mathbf{F}). \\
&= -(\mathbf{p} \wedge \mathbf{W}) + (\mathbf{p} \wedge \mathbf{F}).
\end{aligned}
$$

The first term gives the component of torque $pW\sin\theta$ perpendicular to the paper away from you, which causes the precession of the spin axis around the vertical, and the second term gives the component of torque pF in the plane of the paper as shown, which causes the top to rise and go to sleep.

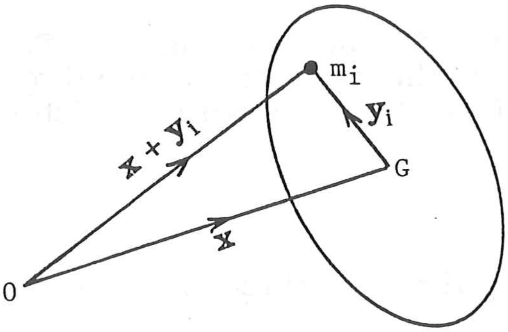
We are now ready to prove Newton's law of spinning motion.

Theorem 3

If a body with centre of gravity G has angular momentum $\mathbf{A}$ about G, and is subjected to a total torque $\mathbf{T}$ about G, then $\dot{\mathbf{A}} = \mathbf{T}$.

Proof

Let $\mathbf{x}$ be the position of G relative to some fixed origin O. Then the position of m_i relative to O is $\mathbf{x} + \mathbf{y}_i$.



Therefore the acceleration of m_i is $\ddot{\mathbf{x}} + \ddot{\mathbf{y}}_i$, by Lemma 3. Let $\mathbf{F}_{ij}$ be the internal force on m_i exerted by m_j. Apply Newton's law of linear motion to m_i:

$$m_i(\ddot{\mathbf{x}} + \ddot{\mathbf{y}}_i) = \mathbf{F}_i + \sum_j \mathbf{F}_{ij}.$$

Take the vector product with $\mathbf{y}_i$ and sum over i:

$$\sum_i \mathbf{y}_i \wedge m_i\ddot{\mathbf{x}} + \sum_i \mathbf{y}_i \wedge m_i\ddot{\mathbf{y}}_i = \sum_i \mathbf{y}_i \wedge \mathbf{F}_i + \sum_{ij} \mathbf{y}_i \wedge \mathbf{F}_{ij}$$

We shall identify each of the four terms in this equation. The first term vanishes because

$$\begin{aligned}\sum \mathbf{y}_i \wedge m_i\ddot{\mathbf{x}} &= \left(\sum m_i\mathbf{y}_i\right) \wedge \ddot{\mathbf{x}}\text{, by Theorem 2 (ii) and (iv)}\\ &= \mathbf{0}\text{, since } \sum m_i\,\mathbf{y}_i = \mathbf{0}.\end{aligned}$$

This is the crucial place where we have used the fact that G is the centre of gravity, and deduced that the acceleration $\ddot{\mathbf{x}}$ of G does not matter.

The second term $= \dot{\mathbf{A}}$ because

$$\begin{aligned}\dot{\mathbf{A}} &= \left(\sum \mathbf{y}_i \wedge m_i\dot{\mathbf{y}}_i\right)^{\boldsymbol{\cdot}}\text{, by definition,}\\ &= \sum (\dot{\mathbf{y}}_i \wedge m_i\dot{\mathbf{y}}_i) + (\mathbf{y}_i \wedge m_i\ddot{\mathbf{y}}_i)\text{, by Theorem 2 (v),}\\ &= \sum \mathbf{y}_i \wedge m_i\ddot{\mathbf{y}}_i\text{, since } \dot{\mathbf{y}}_i \wedge m_i\dot{\mathbf{y}}_i = \mathbf{0} \text{ by Theorem 2 (iii).}\end{aligned}$$

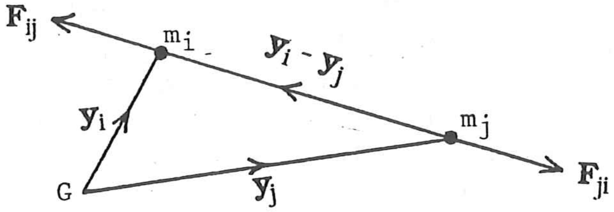
The third term $= \mathbf{T}$, by definition.

Finally the last term vanishes because for each pair $i \neq j$

$$(\mathbf{y}_i \wedge \mathbf{F}_{ij}) + (\mathbf{y}_j \wedge \mathbf{F}_{ji}) = (\mathbf{y}_i \wedge \mathbf{F}_{ij}) + (\mathbf{y}_j \wedge (-\mathbf{F}_{ij})), \quad \text{since } \mathbf{F}_{ji} = -\mathbf{F}_{ij},$$

$$= (\mathbf{y}_i - \mathbf{y}_j) \wedge \mathbf{F}_{ij}, \quad \text{by Theorem 2 (ii) and (iv),}$$

$$= \mathbf{0}, \quad \text{by Theorem 2 (iii) since } \mathbf{F}_{ij} \text{ is parallel to } \mathbf{y}_i - \mathbf{y}_j.$$



Therefore $\dot{\mathbf{A}} = \mathbf{T}$, as required.

Remark

Theorem 3 can be converted back into words as follows.

torque = rate of change of angular momentum

$$= \frac{\text{change in ang. mom.}}{\text{time}}$$

$\therefore$ torque $\times$ time = change in ang. mom.

= (new ang. mom.) - (old ang. mom.)

$\therefore$ new ang. mom. = (old ang. mom.) + (torque $\times$ time).

This completes the proof of Newton's law of spinning motion that was used in the video and Notes 1.

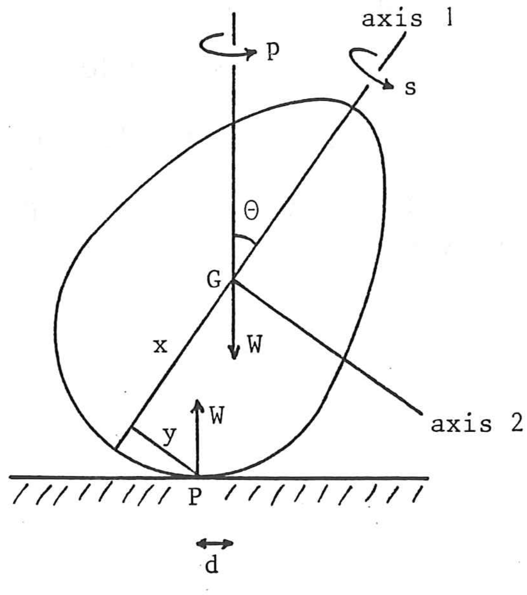
Appendix 2

Why a spinning egg stands on end

The spinning egg is the only experiment in the video that has not yet been dealt with in the book, and an explanation is included here for completeness sake. I remember that in the very first lecture that I ever attended at university, the lecturer aroused our curiosity by spinning an egg on end and then challenged us to explain it. Much to my disappointment it never was explained during the course, and it was not until several years later that I eventually satisfied my curiosity. I hope that in turn the video may have aroused your curiosity, and lest you too be disappointed I thought I had better include an explanation, although it is harder than the other examples.

As explained in Notes 3 the egg is not like a top because the axis of symmetry has fast precession rather than the slow precession observed in tops and wheels. As a result the mathematics is more sophisticated, and this appendix will have to be addressed to the expert rather than the beginner, because I shall assume knowledge of moving axes and moments of inertia that one would normally get in a first or second year university course.

Let W be the weight, and G the centre of gravity of the egg. Choose right-handed orthogonal moving axes at G, so that axis 1 is along the axis of symmetry of the egg (pointing upwards as shown), axis 2 is in the vertical plane through G (pointing downwards as shown), and axis 3 is perpendicular to this plane (pointing away from you).



Let θ = angle between axis 1 and the vertical.
s = spin of the egg about axis 1 in radians/sec.
p = spin of the axes about the vertical in radians/sec due to the precession of the egg.

We shall assume that s is large, and that $\dot{\theta}$ and $\ddot{\theta}$ are small, but that $s\dot{\theta}$ is not necessarily small. We shall ignore small quantities in relation to large quantities if they are added together.

Let Ω = angular velocity of axes = $(p\cos\theta, -p\sin\theta, \dot{\theta})$,
$\mathbf{w}$ = angular velocity of egg = $(s, -p\sin\theta, \dot{\theta})$,
P = point of egg in contact with the table,
$\mathbf{r}$ = position of P relative to G = $(-x, y, 0)$, as shown.

Therefore

$$\dot{\mathbf{r}} = \mathbf{w} \wedge \mathbf{r} = (-y\dot{\theta}, -x\dot{\theta}, sy - px\sin\theta)$$

$$= (0,0,v), \text{ putting } v = sy - px\sin\theta, \text{ and ignoring small } \dot{\theta}.$$

Let $\mathbf{R}$ = reaction of table on egg = $(W\cos\theta, -W\sin\theta, F)$, where F is the frictional force in the direction of axis 3 opposing the sliding of P. Therefore

$$F \lesseqgtr 0 \text{ as } v \gtreqless 0.$$

Let $\mathbf{T}$ = torque about G $= \mathbf{r} \wedge \mathbf{R}$
$= (Fy, Fx, W(x\sin\theta - y\cos\theta))$
$= (Fy, Fx, Wd)$,

where d is the horizontal distance between P and G.

Let a,b = moments of inertia of egg about axes 1,2.

$$\text{Then the inertia tensor } I = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \text{ by symmetry.}$$

Let $\mathbf{A}$ = angular momentum of egg
$= \mathbf{Iw}$
$= (as, -bp\sin\theta, b\dot{\theta})$.

Therefore

$$\dot{\mathbf{A}} = (a\dot{s}, -(bp\sin\theta)^{\boldsymbol{\cdot}}, b\ddot{\theta}).$$

$$\Omega \wedge \mathbf{A} = (0, (as - bp\cos\theta)\dot{\theta}, (as - bp\cos\theta)p\sin\theta).$$

Newton's law of spinning motion with moving axes is:

$$\dot{\mathbf{A}} + \Omega \wedge \mathbf{A} = \mathbf{T}.$$

Here the first term $\dot{\mathbf{A}}$ is the rate of change of angular momentum *relative* to the moving axes, and the second term $\Omega \wedge \mathbf{A}$ is the rate of change *due to* the movement of the axes, and so their sum gives the total rate of change which is equal to $\mathbf{T}$ by Appendix I Theorem 3. Resolving along the axes:

(1) $$a\dot{s} = Fy$$

(2) $$-(bp\sin\theta)^{\cdot} + (as - bp\cos\theta)\dot{\theta} = Fx$$

(3) $$b\ddot{\theta} + (as - bp\cos\theta)p\sin\theta = Wd.$$

Ignore the small $\ddot{\theta}$ in (3), giving

(4) $$(as - bp\cos\theta)p\sin\theta = Wd.$$

This can be regarded as a quadratic equation in p, with a large coefficient of p since s is large. Therefore it has two solutions, one with p large and the other with p small. Initially p is large but we must also consider the case when p is small in order to ensure that the position of the egg spinning on end, given by $\theta = 0$, is stable with respect to both fast and slow precessions.

Case 1: Slow precession. Assume p is small.

Ignore the very small p^2 in (4), giving

$$asp\sin\theta = Wd$$

$$\therefore \quad p = \frac{Wd}{as\sin\theta},$$ confirming that p is small since s is large.

Ignore the small p in the definition of v, giving

$$v = sy$$
$$\therefore v > 0$$
$$\therefore F < 0,$$ since F opposes the sliding of P.

Ignore the small p in (2), giving

$$as\dot{\theta} = Fx.$$

$$\therefore \quad \dot{\theta} = \frac{Fx}{as}$$

$$\therefore \quad \dot{\theta} < 0 \text{ since } F < 0.$$

Therefore θ decreases to zero, proving that spinning on end is stable with respect to slow precession. Note that Case 1 is the simple case, like the sleeping top. Case 2 is the harder case.

Case 2: Fast precession

Write equation (4) as

$$as - bp\cos\theta = \frac{Wd}{p\sin\theta}.$$

Ignore the last term which is small since p is large.

$$as - bp\cos\theta = 0.$$

$\therefore$ $p = \dfrac{as}{b\cos\theta}$, confirming that p is large since s is large.

$\therefore$ $v = sy - px\sin\theta$

$= sy - \dfrac{asx\tan\theta}{b}$, substituting the value of p,

$= \dfrac{s}{b}(by - ax\tan\theta)$.

We shall show in a geometric Lemma below that

$$ax\tan\theta > by$$

$\therefore$ $v < 0$

$\therefore$ $F > 0$ since F opposes the sliding of P.

Substitute the value of p in (2), giving

$$-(as\tan\theta)^{\cdot} = Fx.$$

$\therefore$ $-a\dot{s}\tan\theta - as\dot{\theta}\sec^2\theta = Fx.$

Substituting (1) gives

$$-Fy\tan\theta - as\dot{\theta}\sec^2\theta = Fx$$

$\therefore$ $as\dot{\theta}\sec^2\theta = -F(x + y\tan\theta)$

$\therefore$ $\dot{\theta} = -\dfrac{F\cos^2\theta(x + y\tan\theta)}{as}$

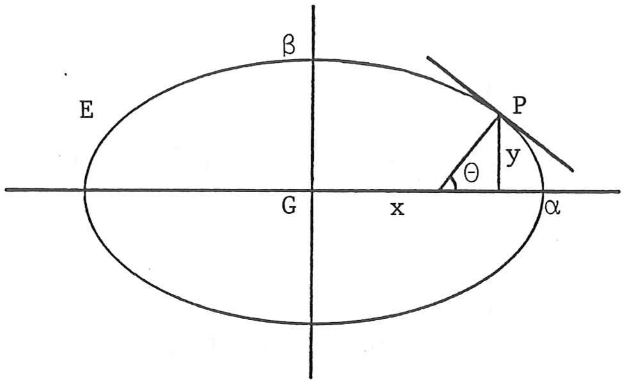
$\therefore$ $\dot{\theta} < 0$ because $F > 0$.

Therefore θ decreases to zero as in the last case, even though the direction of friction has reversed. Therefore the egg rises to spin on end, and that position is stable with respect to both fast and slow precession.

There remains to prove the geometric lemma. Let E be the ellipse with semi-axes α, β given by

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = 1,$$

where $\alpha > \beta$. We shall approximate the shape of the egg by the ellipsoid of revolution given by revolving E around the x-axis.



Let P = (x,y) be a point on E, and let θ be the angle between the normal at P and the x-axis. Notice that the diagram is the same as that of the spinning egg with its axis of symmetry drawn horizontal.

Lemma

$$ax\tan\theta > by.$$

Proof

Let m be the mass of the egg. Then the moments of inertia of the egg about the x,y axes are given by

$$a = \frac{2}{5}m\beta^2 \quad , \quad b = \frac{1}{5}m(\alpha^2 + \beta^2).$$

$$\therefore \quad \frac{b}{a} = \frac{\alpha^2+\beta^2}{2\beta^2} < \frac{2\alpha^2}{2\beta^2} = \frac{\alpha^2}{\beta^2}.$$

If we parametrise P by $x = \alpha\cos\varphi$, $y = \beta\sin\varphi$, then

$$\text{slope of tangent} = -\frac{\beta\cos\varphi}{\alpha\sin\varphi} = -\frac{\beta^2 x}{\alpha^2 y}$$

$$\therefore \quad \text{slope of normal} = \tan\theta = \frac{\alpha^2 y}{\beta^2 x} > \frac{b}{a}\cdot\frac{y}{x}$$

$\therefore$ $ax\tan\theta > by$, as required.

Spinning smarties

Smarties are small ellipsoidal chocolate sweets that are not only good to eat, but also good for spinning. If you spin a smartie horizontally about its axis of symmetry then surprisingly it will rise up and spin on its edge. Paradoxically as in the egg the stable and unstable positions of stationary equilibrium are reversed when spinning.

The mathematics is exactly the same as that for an egg with one exception, as follows. A smartie is a flattened ellipsoid, whereas an egg is an elongated ellipsoid. In other words if α denotes the length of the semi-axis along the axis of symmetry, and β the lengths of the other two semi-axes, then in a smartie $\alpha < \beta$, whereas in an egg $\alpha > \beta$. Therefore the inequality in the proof of the geometric lemma above is reversed, $b/a > \alpha^2/\beta^2$, and so the geometric lemma is reversed, $ax\tan\theta < by$. Therefore in Case (2) of the fast precession the direction of v is reversed, F is reversed, and so $\dot{\theta} > 0$. Therefore a horizontally spinning smartie is unstable with respect to *fast* precession (although it remains stable with respect to *slow* precession). Consequently any slight peturbation will be automatically magnified, causing it to rise up and spin on edge. Furthermore one can show that spinning on edge is stable provided that the spin is sufficiently fast; when the spin slows down it will reach a point where the smartie will suddenly topple over like a sleeping top waking up.

Recommended further reading

1. G.R. Fowles, *Analytical mechanics*, CBS College Publishing, 1986.

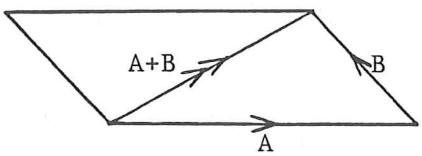
2. A. Gray, *A treatise on gyrostatics and rotational motion*, 1918 (reprinted Dover Publications, 1959).

Gyroscopes and boomerangs

Solutions 1

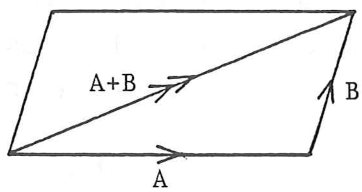
ALL MEASUREMENTS IN CENTIMETRES

1. (i)



A = 4, B = 2, A+B = 3

(ii)



A = 4, B = 2, A+B = 5

2. XB = 7. Torque = 7F. Torque-axis away from you.
   XA = 3. New torque = $3 \times 2F = 6F$, smaller.

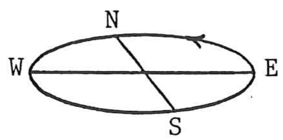
3. AX = 2, XB = 5, AB = 7.
   (i) 2F (ii) 5F (iii) 7F. Torque axis away from you.
   AY = 4, YB = 3.
   (i) 4F (ii) 3F (iii) 7F.

4. 5F away, 5F towards, 3F away, 4F towards.

5. dF away.

6.

position	1	2	3	4
spin axis	E	N	W	S
torque axis	N	W	S	E



7. $2\pi r$ cm; $2\pi rs$ cm; $2\pi rs$ cm/sec, 462 cm/sec.

8. 11 sec, 7 sec, 5 sec.

9. None. It turns out that a change of 2% in g does not affect the answer, because the calculations were only to the nearest second. It would be misleading to make the calculations more accurate because the other measurements are only accurate to within 10%.

Gyroscopes and boomerangs

Solutions 2

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1. $\theta = 37^o$ $\quad r_1 = 2.5$, $x_1 = 2$, $y_1 = 1.5$
$r_2 = 3.5$, $x_2 = 2.8$, $y_2 = 2.1$
$r_3 = 5$, $x_3 = 4$, $y_3 = 3$.

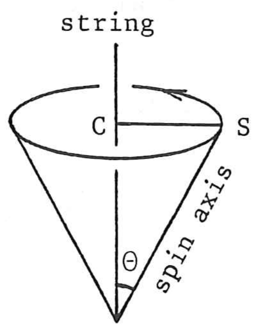
$$\frac{2}{2.5} = \frac{2.8}{3.5} = \frac{4}{5} = \cos 37^\circ , \quad \frac{1.5}{2.5} = \frac{2.1}{3.5} = \frac{3}{5} = \sin 37^\circ .$$

2. (i) $\theta = 60^o$, $r = 4$, $r\sin\theta = 3.5$
(ii) $\theta = 45^o$, $r = 4$, $r\sin\theta = 2.8$
(iii) $\theta = 30^o$, $r = 4$, $r\sin\theta = 2$

3. $\theta = 30^o$, $d = 4$.
Distance between lines of action of forces = $d\sin\theta = 2$.
Torque = $(d\sin\theta)F = 2F$.

4. $T_o = dW$
$T = (d\sin\theta)W = dW\sin\theta = T_o\sin\theta$.

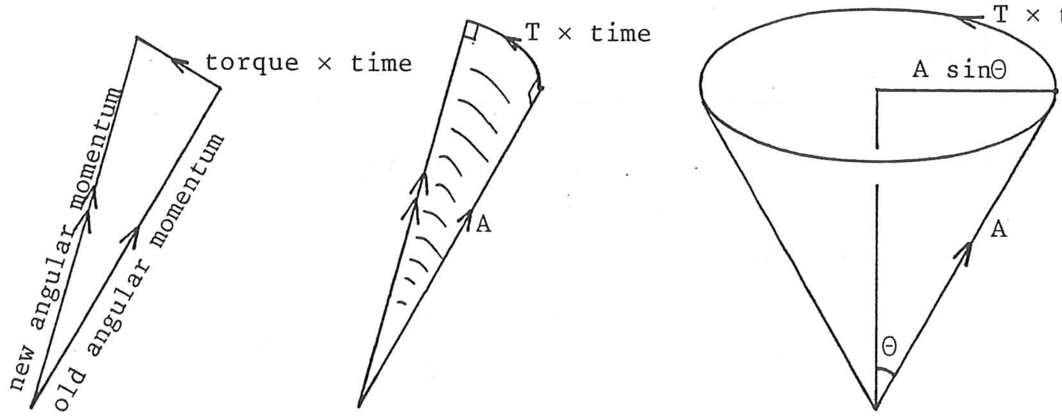
The plane containing the forces is the plane of the paper and therefore the torque axis is horizontal and perpendicular to the paper. By the right hand screw rule it is away from you.



Let S be a point on the spin axis, and let C be the point on the string at the same height as S . By the gyro law S begins to move horizontally away from you at right-angles to CS . Therefore S precesses in the horizontal circle centre C . Therefore the spin axis precesses in a cone at constant angle θ to the vertical.

5. Newton's law of spinning motion gives the triangle of vectors

new angular momentum = (old angular momentum) + (torque × time).



When this triangle is modified to take account of the fact that the torque axis is always horizontal and at right angles to the spin axis, then the triangle becomes a sector of the surface of a cone, with straight sides of length equal to the angular momentum A, and a curved arc at the top of length equal to torque × time. After time t the angular momentum vector will have rotated once round keeping at constant angle θ to the vertical, and will have traced out the complete cone. The base of the cone (which is at the top because the cone is upside down) is a circle with circumference T × t and radius Asinθ.

$$\therefore\ T \times t = 2\pi(A\sin\theta)$$

$$\therefore\quad t = \frac{2\pi A\sin\theta}{T}$$

$$= \frac{2\pi A\sin\theta}{T_o\sin\theta}, \qquad \text{substituting } T = T_o\sin\theta,$$

$$= \frac{2\pi A}{T_o}, \quad \text{cancelling } \sin\theta \text{ from top and bottom.}$$

Therefore t is independent of θ, because both A and T_o are.

6. If the upthrust were at the tip, then the distance between the lines of action of the forces would be dsinθ.

$$\therefore\ \text{torque} = (d\sin\theta)W = Wd\sin\theta\ .$$

As in Question 4 the torque axis is horizontal, perpendicular to the paper and away from you. As in Question 4 S precesses in a horizontal circle with centre vertically above G. Since SGP is a straight line, and G is stationary, P must precess in a similar circle with centre vertically below G. Therefore the spin axis precesses in double cone with vertex G, at constant angle θ to the vertical.

7. Due to the spin P travels a distance $2\pi r$ in one revolution, and $2\pi rs$ in one second.

$$\therefore\ v_1 = 2\pi rs.$$

Due to precession P travels a distance $2\pi R$ in one circle of precession, and $2\pi Rp$ in one second.

$$\therefore\ v_2 = 2\pi Rp.$$

If s increases then the angular momentum A increases, the time t to precess once round increases because A is on top of the formula for t (see Question 5), and hence the rate of precession p decreases. Therefore v_1 increases and v_2 decreases until $v_1 > v_2$. Therefore if s is sufficiently large then $v > 0$.

The second torque $= GP \times F$, since GP is perpendicular to F,
$= dF$, approximately.

The second torque axis is perpendicular to the plane Q containing G and the line of action of F. Now F is perpendicular to the paper, and so Q is also. Therefore the second torque axis lies in the plane of the paper. It is also perpendicular to GP since Q contains GP. Therefore the second torque axis is perpendicular to the spin axis, approximately. By the right-hand screw rule it points in the upwards direction.

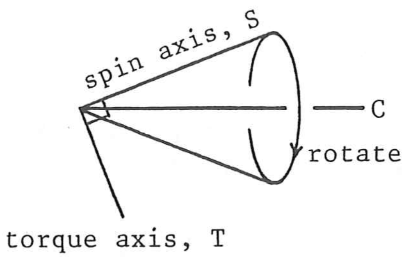
By the gyro law the spin axis chases the second torque axis, and therefore moves towards the vertical. Combining this with the precession gives an upward spiral towards the vertical. Hence the top goes to sleep.

8. When the top slows down s decreases, p increases, v_1 drops below v_2, v changes from positive to negative, F reverses direction, the second torque axis reverses direction, and so spin axis chases it downwards instead of upwards. Therefore the sleeping top wakes up, the spin axis spirals outwards until $v_1 = v_2$, and then the top rolls noisily for a few moments while precessing at a constant angle to the vertical. Finally v goes negative again and the top topples right over.

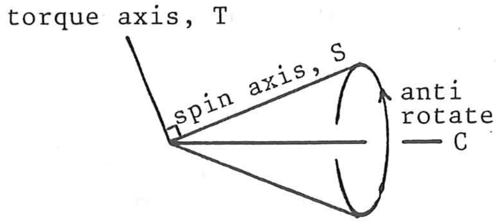
Gyroscopes and Boomerangs

Solutions 3

1. Easy



Hard



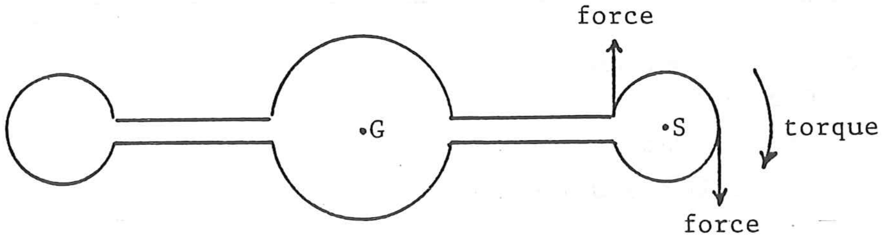
Let S denote the spin axis, and C the axis of the cone. If you try and rotate S around C in the same direction as the spin you will in fact be applying a torque whose axis T is at right-angles to S and points inwards towards C. By the gyro law S will move towards T, and appear to be attracted towards C, thereby helping you to rotate it around the cone.

Conversely if you try and rotate S in the opposite direction then T will point outwards away from C. By gyro law S will move towards T and away from the cone; it will appear to be repelled by C and will resist going the way you want it to. If you then try and bring S back towards the cone then you will in fact be applying a torque whose axis points in the same direction as you were originally trying to move S. Therefore by the gyro law S will begin to move in that direction, and to your surprise the wheel will suddenly appear to cooperate with you instead of resisting you, causing it to wildly overshoot in that direction. The more you try to bring it under control the more perverse and wild it will appear to become.

2. The plane containing the forces is the plane of the paper, and so the torque axis is perpendicular to the paper. By the right-hand screw rule it is towards you. Therefore S precesses in a cone around E at a constant angle to E, similar to the cases of a wheel on a string or a top without friction (Worksheet 2 Questions 5 and 6). The only difference here is that the precession is in the opposite direction to the spin.

3. Similar to the last question.

4. The forces exerted on the space-ship are equal and opposite to the forces that the space-ship uses to squirt out the jets.



Therefore the torque is clockwise and so the space-ship begins to rotate clockwise. The sum of the forces is zero and so G remains stationary (by Appendix 1 Example 3). Therefore the space-ship rotates about G.

Gyroscopes and Boomerangs

Solutions 4

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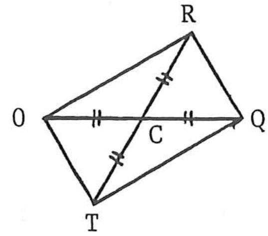
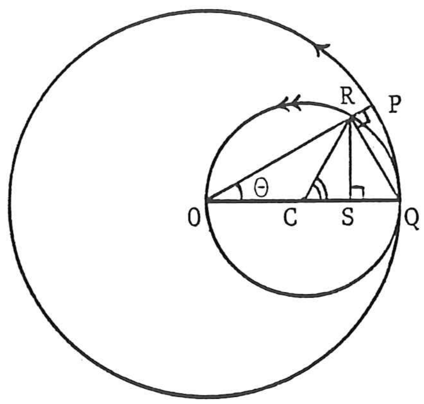
1. $x = \cos\theta$, $y = \sin\theta$.
   $OQ = 1$
   $\therefore\ OR = \cos\theta$
   $\therefore\ OS = OR\cos\theta = \cos^2\theta.$



Complete the rectangle ORQT. Let C be the intersection of the diagonals. By symmetry CO = CR = CQ = CT. Therefore the circle centre C radius CO goes through the four corners. But this is the circle diameter OQ, which therefore goes through R.

$$C\hat{R}O = C\hat{O}R, \text{ since } CO = CR,$$
$$= \theta\ .$$

$$\therefore\ S\hat{C}R = C\hat{O}R + C\hat{R}O\ , \text{ from triangle COR,}$$
$$= 2\theta.$$



As P goes steadily round the larger circle, R goes steadily twice round the smaller circle. Therefore by symmetry the average value of OS is OC. But OC = 1/2. Therefore the average value of $\cos^2\theta$ is 1/2.

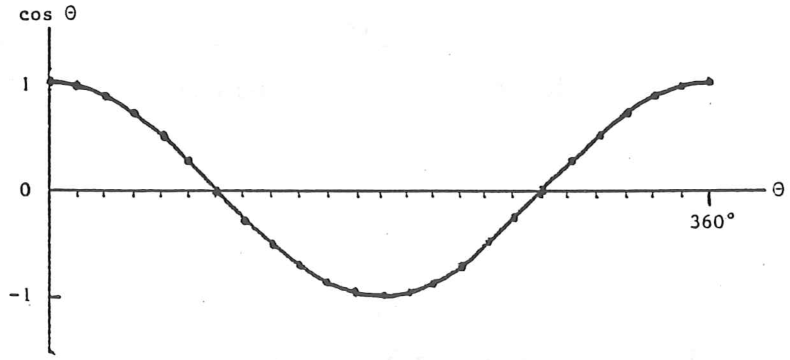
By the symmetry of interchanging x and y the average value of $\sin\theta$ is the same as that of $\cos\theta$, namely zero. Similarly the average value of $\sin^2\theta$ is the same as that of $\cos^2\theta$, namely 1/2.

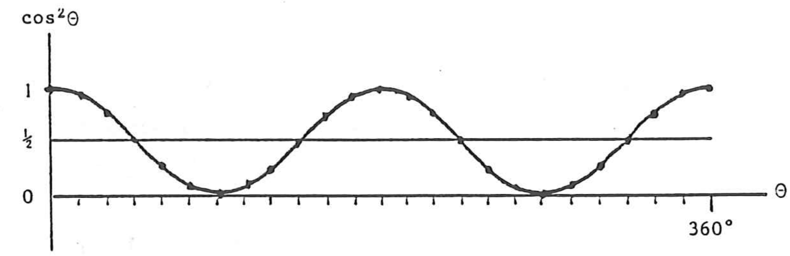
$$RS = OR\sin\theta = \cos\theta\sin\theta.$$

Hence by symmetry as R goes round the smaller circle the average value of $\cos\theta\sin\theta$ is zero.

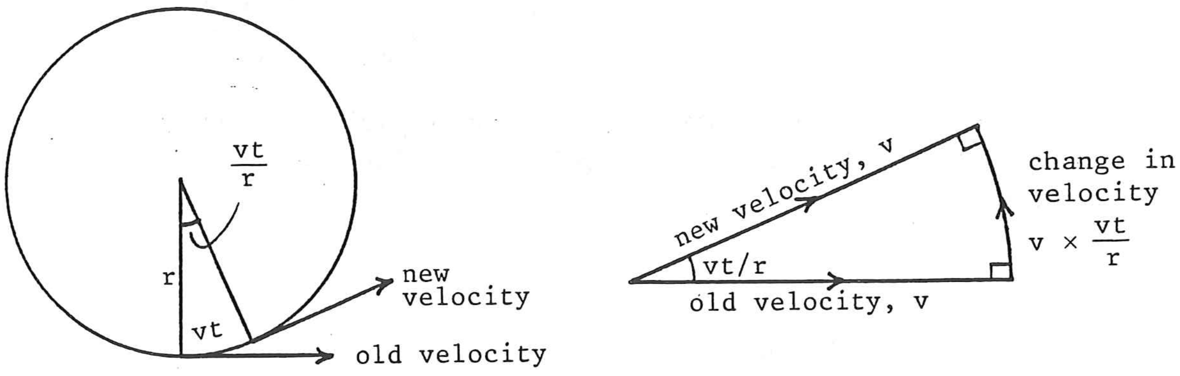
2.

θ	cosθ	cos²θ
0	1	1
15	0.96	0.93
30	0.87	0.75
45	0.71	0.5
60	0.5	0.25
75	0.26	0.07
90	0	0
105	-0.26	0.07
120	-0.5	0.25
135	-0.71	0.5
150	-0.87	0.75
165	-0.96	0.93
180	-1	1
195	-0.96	0.93
210	-0.87	0.75
225	-0.71	0.5
240	-0.5	0.25
255	-0.26	0.07
270	0	0
285	0.26	0.07
300	0.5	0.25
315	0.71	0.5
330	0.87	0.75
345	0.96	0.93
360	1	1





3. In a small time t the mass will have travelled a distance vt through an angle vt/r radians.

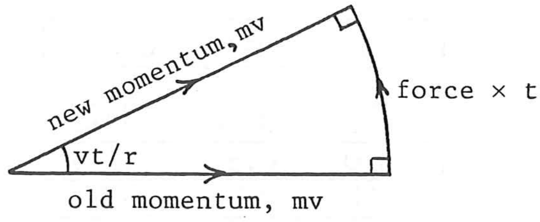


$\therefore$ change in velocity $= \dfrac{v^2 t}{r}$ towards O, approximately.

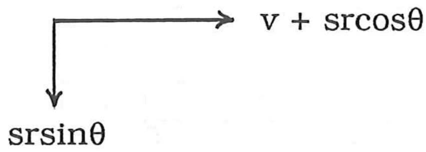
$\therefore$ acceleration $= \dfrac{\text{change in velocity}}{t} = \dfrac{v^2}{r}$ towards O.

By Newton's law of linear motion

$$\begin{aligned} \text{force} \times t &= \text{change in momentum} \\ &= mv \times \frac{vt}{r} \text{ towards O, approximately.} \\ \therefore \quad \text{force} &= \frac{mv^2}{r} \text{ towards O.} \end{aligned}$$



4. P has components of velocity:



$$\begin{aligned} \therefore \ (\text{speed of P})^2 &= (v + sr\cos\theta)^2 + (sr\sin\theta)^2 \\ &= v^2 + 2vsr\cos\theta + s^2r^2\cos^2\theta + s^2r^2\sin^2\theta \\ &= v^2 + s^2r^2 + 2vsr\cos\theta, \text{ since } \cos^2\theta + \sin^2\theta = 1. \end{aligned}$$

$$\frac{\text{Aerodynamic factor of B}}{\lambda} = \frac{\text{length of B}}{\text{length of blades}} = \frac{\delta}{nb}$$

$$\therefore \text{ aerodynamic factor of B} = \frac{\lambda\delta}{nb}.$$

$$\begin{aligned} \text{Lift generated by B} &= (v^2 + s^2r^2 + 2vsr\cos\theta)\frac{\lambda\delta}{nb} \\ &= (v^2 + s^2r^2)\frac{\lambda\delta}{nb} + \frac{2vsr\lambda\delta}{nb}\cos\theta. \end{aligned}$$

The first term is constant, and so equals its average, while the second term has average zero because the average of cosθ is zero by Question 1.

$$\begin{aligned} L &= n\int_0^b (v^2 + s^2r^2)\frac{\lambda}{nb}\,dr \\ &= \frac{\lambda v^2}{b}\int_0^b dr + \frac{\lambda s^2}{b}\int_0^b r^2\,dr \\ &= \frac{\lambda v^2}{b}b + \frac{\lambda s^2}{b}\cdot\frac{b^3}{3} \\ &= \lambda v^2\left[1 + \frac{1}{3}\left(\frac{sb}{v}\right)^2\right]. \end{aligned}$$

$2v/3$ = forward speed of end of lower blade = $v - sb$

$$\therefore \quad sb = v - \frac{2}{3}v = \frac{1}{3}v$$

$$\therefore \quad \frac{sb}{v} = \frac{1}{3}$$

$$\therefore \frac{1}{3}\left(\frac{sb}{v}\right)^2 = \frac{1}{27}.$$

Ignoring $1/27$ compared with 1 gives $L = \lambda v^2$.

But

$$L = \frac{mv^2}{R}$$

$$\therefore \lambda v^2 = \frac{mv^2}{R}$$

$$\therefore \quad R = \frac{m}{\lambda}$$

$$\therefore \quad \Omega_1 = \frac{v}{R} = v \times \frac{\lambda}{m} = \frac{v\lambda}{m} \qquad \text{radians/sec.}$$

T_B = distance × force

$$= r \times (v^2 + s^2r^2 + 2vsr\cos\theta)\,\frac{\lambda\delta}{nb}.$$

$\therefore$ component along trailing axis $= T_B\cos\theta$

$$= r[(v^2 + s^2r^2)\cos\theta + 2vsr\cos^2\theta]\,\frac{\lambda\delta}{nb}$$

$\therefore$ average $= \dfrac{vsr^2\lambda\delta}{nb}$, because the average of $\cos\theta$ is zero and the average of $\cos^2\theta$ is $1/2$ by Question 1.

Meanwhile the component perpendicular to the trailing axis

$$= T_B\sin\theta$$

$$= r[(v^2 + s^2r^2)\sin\theta + 2vsr\cos\theta\sin\theta]\,\frac{\lambda\delta}{nb}$$

$$\therefore \text{average} = 0$$

because the average of each of $\sin\theta$ and $\cos\theta\sin\theta$ is zero by Question 1. Therefore the total torque T about G is in the direction of the trailing axis, and

$$T = n\int_0^b \frac{vsr^2\lambda}{nb}dr$$
$$= \frac{vs\lambda}{b}\int_0^b r^2\,dr$$
$$= \frac{vs\lambda}{b}\frac{b^3}{3}$$
$$= \frac{1}{3}vsb^2\lambda.$$

$$\frac{\text{Mass of B}}{m} = \frac{\text{length of B}}{\text{length of blades}} = \frac{\delta}{nb}$$

$$\therefore \text{ mass of B} = \frac{m\delta}{nb}$$

$$\text{Speed of B relative to G} = sr$$

$$\therefore \text{ momentum of B relative to G} = \frac{m\delta}{nb} \times sr$$

$$\therefore \text{ angular momentum of B about G} = r \times \frac{m\delta}{nb} \times sr = \frac{msr^2\delta}{nb}$$

$$\therefore \text{ total angular momentum A} = n\int_0^b \frac{msr^2}{nb}dr$$
$$= \frac{ms}{b}\int_0^b r^2\,dr$$
$$= \frac{1}{3}msb^2$$

By Newton's law of spinning motion,

$$\Omega_2 = \frac{T}{A} \text{ radians/sec.}$$

[Note that it is correct to calculate the angular momentum A about the centre of gravity G, even though G itself may be moving, by Appendix 1 Theorem 3.]

$$\therefore \Omega_2 = \frac{\frac{1}{3}vsb^2\lambda}{\frac{1}{3}msb^2} = \frac{v\lambda}{m} = \Omega_1 \text{, as required.}$$