GEOMETRIC UNFOLDING OF A DIFFERENCE EQUATION

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PROBLEM (Gerry Ladas)

Describe the behaviour of the sequence of positive real numbers

$$S = \{x_1, x_2, x_3, \dots\}$$

generated by the equation

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \quad n \geq 2,$$

where $\begin{cases} \alpha \geq 0 & \text{is a given constant, and} \\ x_1, x_2 > 0 & \text{are given initial terms.} \end{cases}$

Example 1 $\boxed{\alpha = 1}$

Let $x_1 = 1$

$x_2 = 1$

Then $x_3 = \frac{1+1}{1} = 2$

$x_4 = \frac{1+2}{1} = 3$

$x_5 = \frac{1+3}{2} = 2$

$x_6 = \frac{1+2}{3} = 1$

$x_7 = \frac{1+1}{2} = 1.$

$\therefore S = \{\underbrace{1, 1, 2, 3, 2}_{\text{periodic with period 5}}, 1, 1, \dots\}$

Example 2 $\boxed{\alpha=2}$

Let $x_1 = 2$

$x_2 = 2$

Then $x_3 = \frac{2+2}{2} = 2$

$\therefore S = \{2, 2, 2, 2, 2, \cdots\}$

Constant

Similarly $\alpha = u^2 - u$

$S = \{u, u, u, \cdots\}$

Example 3 $\boxed{\alpha = 3}$

Let $x_1 = 3$

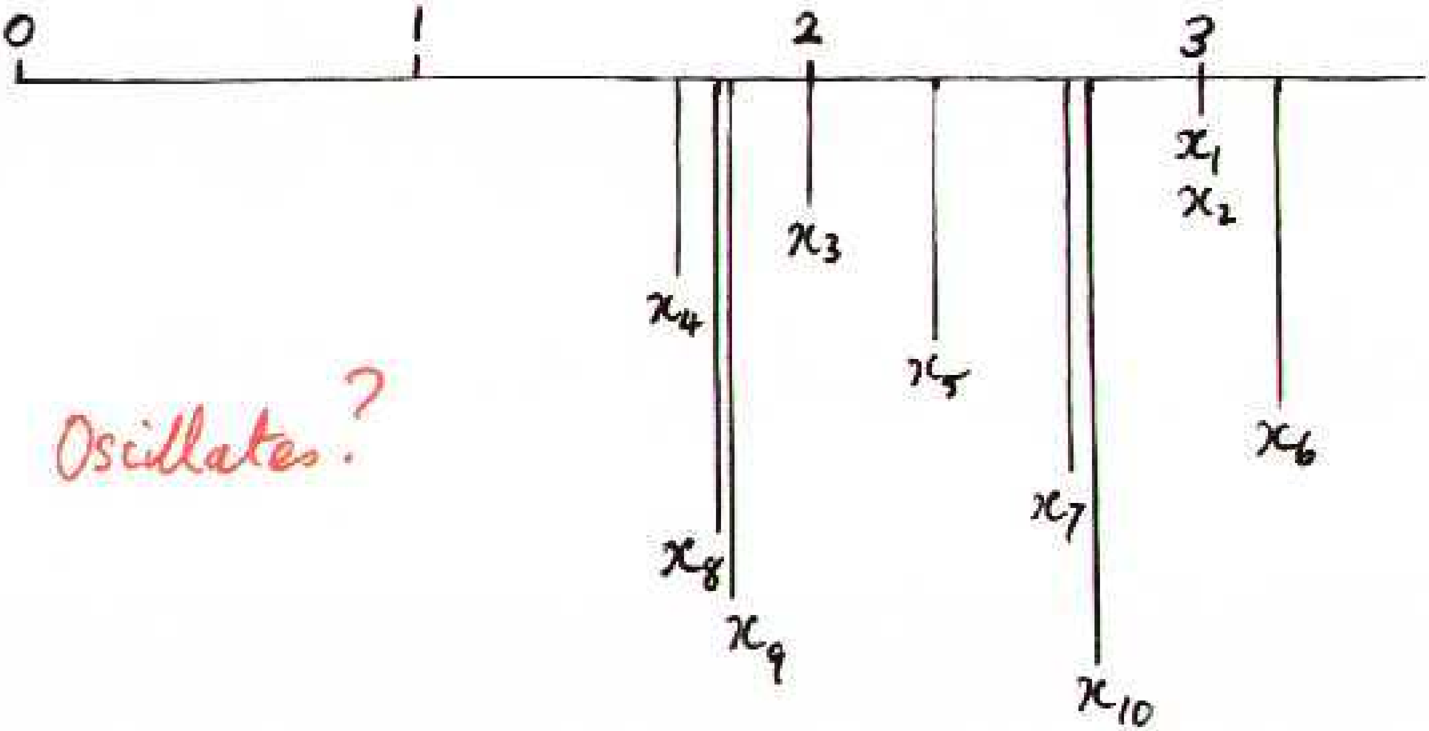
$x_2 = 3$

Then $x_3 = \dfrac{3+3}{3} = 2$

$x_4 = \dfrac{3+2}{3} = \dfrac{5}{3}$

$\vdots$

$$S = \left\{3, 3, 2, \frac{5}{3}, \frac{7}{3}, \frac{16}{5}, \frac{93}{35}, \frac{99}{56}, \frac{445}{248}, \frac{8323}{3069}, \cdots\right\}$$



Oscillates?

THEOREM 1 (Lyness 1942)

If $\alpha = 1$ then S is 5-periodic.

Proof Let $x_1 = x \quad x_2 = y$

Then $x_3 = \dfrac{1+y}{x} \qquad x_4 = \dfrac{1+\frac{1+y}{x}}{y} = \dfrac{1+x+y}{xy}$

$$x_5 = \frac{1+\frac{1+x+y}{xy}}{\frac{1+y}{x}} = \frac{1+x+y+xy}{y(1+y)} = \frac{(1+x)\cancel{(1+y)}}{y\cancel{(1+y)}} = \frac{1+x}{y}$$

$$x_6 = \frac{1+\frac{1+x}{y}}{\frac{1+x+y}{xy}} = \frac{\cancel{(1+x+y)}x}{\cancel{1+x+y}} = x$$

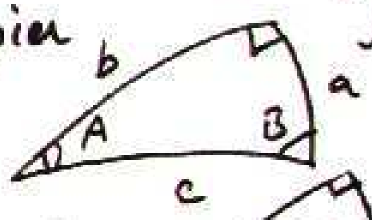
$$x_7 = \frac{1+x}{\frac{1+x}{y}} = y$$

$$\therefore S = \{\underbrace{x, y, \frac{1+y}{x}, \frac{1+x+y}{xy}, \frac{1+x}{y}}_{\text{5-periodic.}}, x, y, \dots\}$$

HISTORY OF THE 5-CYCLE

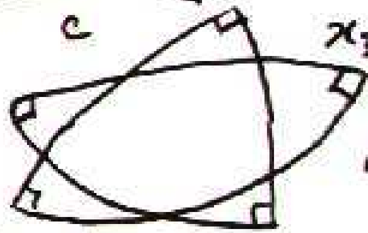
$$\boxed{x_{n+1} = \frac{1+x_n}{x_{n-1}}}$$

1602 Nathaniel Torporley } spherical right-angle triangles
1614 Napier



$x_1 = -\sin^2 A$
$x_2 = -\sin^2 B$
$x_3 = -\cos^2 b$
$x_4 = -\sin^2 c$
$x_5 = -\cos^2 a$

1818 Gauss



Pentagramma mirificum

~1850 Lobachevsky } hyperbolic right-angle triangles
~1860 Schläfli

1942 Lyness : { recurrence formula
{ 3 integers whose sums & differences are squares.

1961 Sawyer : cross-ratios $x_1 = (ADBC)$ $x_2 = (BECD)$
$x_3 = (CADE)$ $x_4 = (DBEA)$ $x_5 = (ECAB)$

1971 Coxeter : frieze patterns Acta Arith. 18 (1971) 297–310.

$$\begin{array}{ccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
& x_1 & & x_3 & & x_5 & & x_2 & & x_4 & & x_1 & & x_3 & & x_5 & & & \\
x_5 & & x_2 & & x_4 & & x_1 & & x_3 & & x_5 & & x_2 & & x_4 & & x_1 & & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

Each diamond $\begin{array}{ccc} & b & \\ a & \diamond & d \\ & c & \end{array}$ satisfies $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$

LEMMA

If $\alpha = 0$ then S is 6-periodic

PROOF

$$x_{n+1} = \frac{x_n}{x_{n-1}}$$

$$\therefore S = \Big\{ \underbrace{x, y, \frac{y}{x}, \frac{1}{x}, \frac{1}{y}, \frac{x}{y}}_{\text{6-periodic}}, x, y, \ldots \Big\}$$

LEMMA

If $\alpha = \infty$ then S is 4-periodic

Proof

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}$$

Change variable $x_n = \sqrt{\alpha}\, y_n$.

$$\therefore \sqrt{\alpha}\, y_{n+1} = \frac{\alpha + \sqrt{\alpha}\, y_n}{\sqrt{\alpha}\, y_{n-1}}$$

$$\therefore \quad y_{n+1} = \frac{1 + \frac{1}{\sqrt{\alpha}} y_n}{y_{n-1}}$$

Put $\alpha = \infty$ $\therefore \frac{1}{\sqrt{\alpha}} = 0$.

$$\therefore y_{n+1} = \frac{1}{y_{n-1}}$$

$$\therefore S = \{\underbrace{x, y, \tfrac{1}{x}, \tfrac{1}{y}}_{\text{4-periodic}}, x, y, \ldots\}$$

SUMMARY

α	0	$0<\alpha<1$	1	$1<\alpha<\infty$	∞
period	6	?	5	?	4

Question

How to tackle these ?

Answer

Unfold the problem.

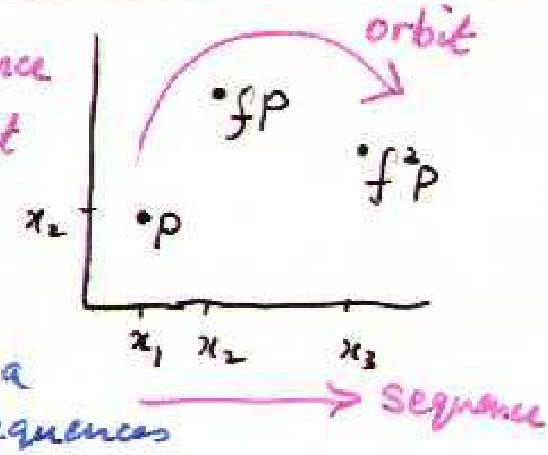
Unfolding of the equation is the diffeomorphism

$$f : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+^2$$
$$(x,y) \longmapsto \left(y, \frac{\alpha+y}{x}\right)$$

Unfolding of the sequence S $(S = \{x_1, x_2, x_3, \dots\})$

$$= \{(x_1,x_2),(x_2,x_3),(x_3,x_4),\dots\} \subset \mathbb{R}_+^2$$
$$= \{P, fP, f^2P, \dots\}$$
$$= \text{the orbit of } P \text{ under } f.$$

Thus $\begin{cases} \text{the orbit} = \text{unfolding of the sequence} \\ \text{the sequence} = \text{projection of the orbit} \end{cases}$



The great advantage of the unfolding is that the orbits can be visualised, and handled as a dynamical system (whereas the sequences are all self-entangled).

THEOREM 2 (Lyness 1945, Ladas 1995)

$$V = \frac{(x+1)(y+1)(x+y+\alpha)}{xy} : \mathbb{R}_+^2 \to \mathbb{R} \text{ is an invariant of } f.$$

COROLLARY

Each orbit lies on a level-curve $V =$ constant.

PROOF We have to show $Vf = V$.

$$Vf(x,y) = V\left(y, \frac{\alpha+y}{x}\right)$$

$$= \frac{(y+1)\left(\frac{\alpha+y}{x}+1\right)\left(y+\frac{\alpha+y}{x}+\alpha\right)}{y\left(\frac{\alpha+y}{x}\right)}$$

$$= \frac{(y+1)(\alpha+y+x)(xy+\alpha+y+\alpha x)}{xy(\alpha+y)}$$

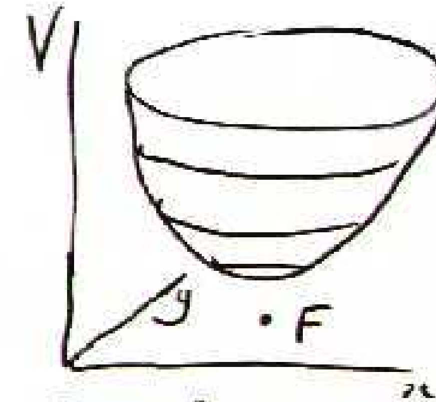
$$= \frac{(y+1)(x+y+\alpha)(x+1)\cancel{(\alpha+y)}}{xy\cancel{(\alpha+y)}}$$

$$= V(x,y)$$

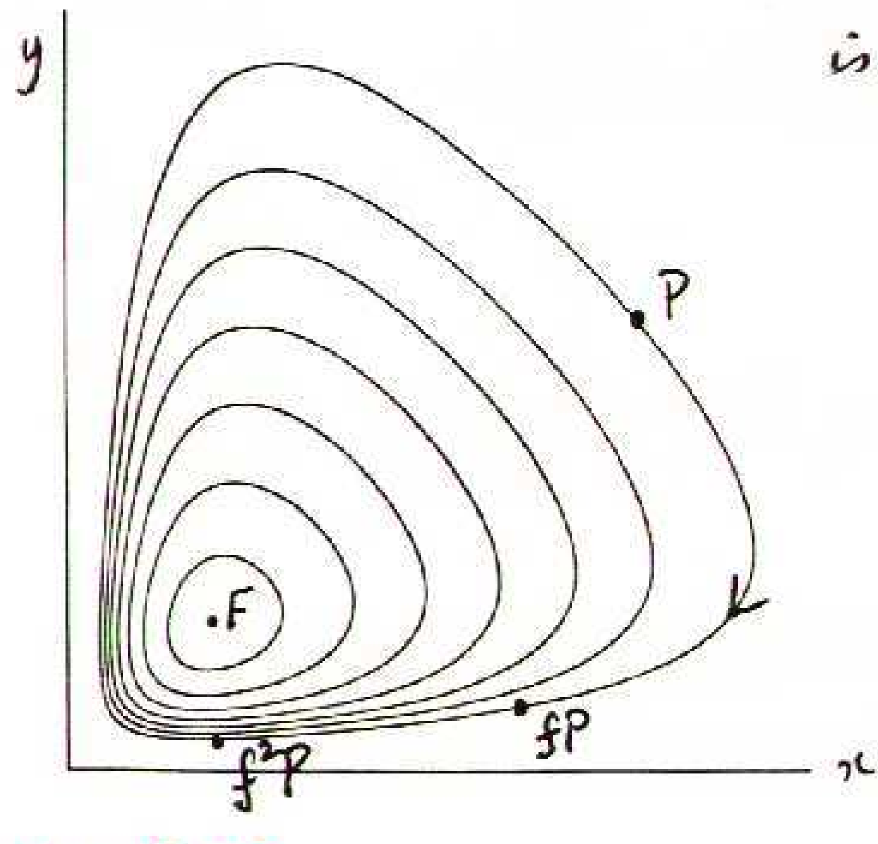
THEOREM 3 The graph of V is bowl-shaped

with a minimum at $F = (\omega, \omega)$

where $\omega = \frac{1 + \sqrt{1+4a}}{2}$



COROLLARY 1. The level-curves of V form a nested family of closed curves encircling F and filling $\mathbb{R}^2_+$. The orbits of f march round these closed curves. Therefore f is a twist-map.



Computer drawing by Mary Lou Zeeman

COROLLARY 2

All the sequences are bounded

ROTATION NUMBERS

Let S^1 denote the unit circle.

Let $r_\rho : S^1 \to S^1$ denote Euclidean rotation through angle $2\pi\rho$, where $0 \le \rho < 1$.

DEFINITION.

If C is a closed curve then a homeomorphism $\varphi : C \to C$ is called <u>rotation-like</u> if it is conjugate to a rotation; in other words if $\exists$ homeomorphism $h : S^1 \to C$ such that the diagram commutes

$$\begin{tikzcd} S^1 \arrow[r, "r_\rho"] \arrow[d, "h"'] & S^1 \arrow[d, "h"] \\ C \arrow[r, "\varphi"] & C \end{tikzcd}$$

Then ρ is called the <u>rotation number</u> of φ.

REMARK

All homeos have a rotation-number (with a more general definition), but most homeos are <u>not</u> rotation-like (i.e. are <u>not</u> conjugate to a rotation of a circle).

THEOREM 4

If C is a level-curve of V then $f|C$ is rotation-like.

(Proof uses algebraic geometry & elliptic curves)

COROLLARY 1

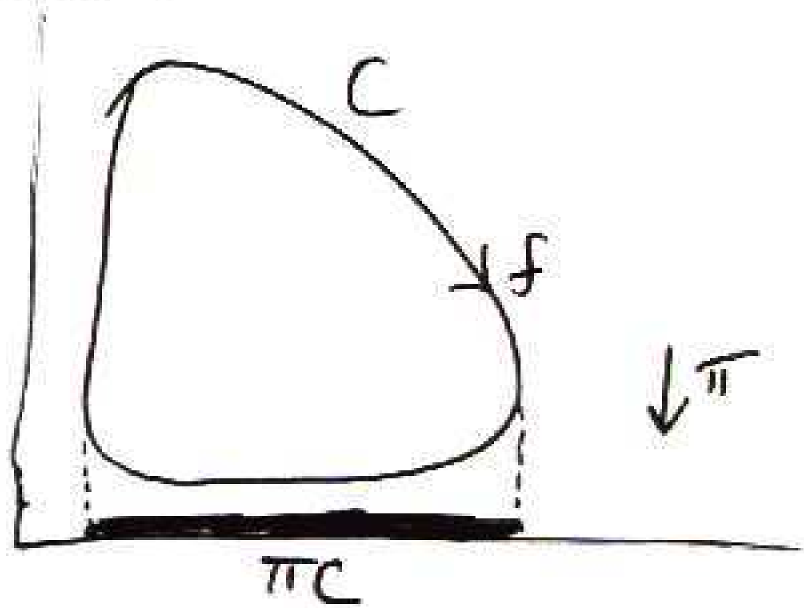
Let ρ = rotation number of $f|C$.

If ρ = rational $\frac{p}{q}$ then each orbit on C is q-periodic.

If ρ = irrational then each orbit on C is dense in C

COROLLARY 2

The corresponding sequences are q-periodic or dense in the interval πC



NOTATION

Given α, $0<\alpha<\infty$, let f_α denote the unfolding

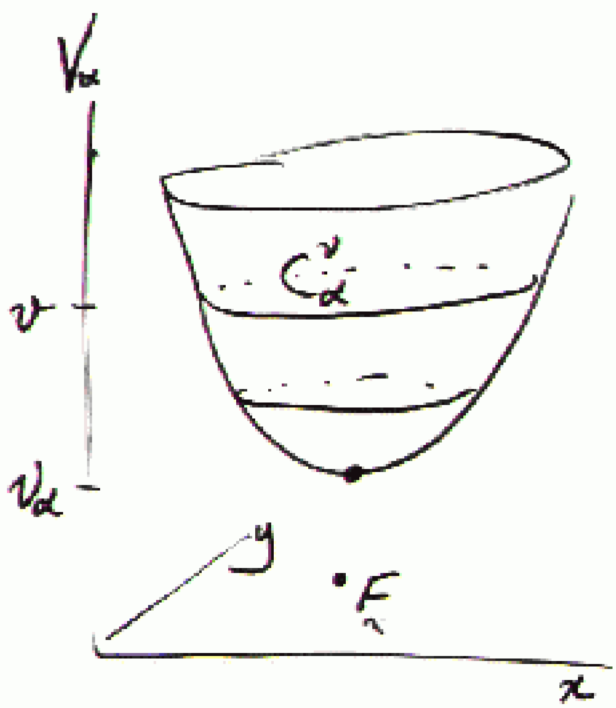
V_α " " invariant function

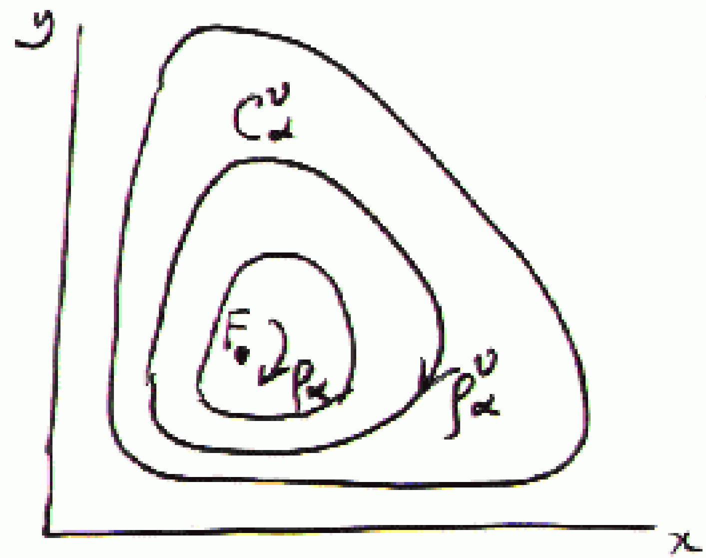
$v_\alpha = \min V_\alpha$.

Given v, $v_\alpha < v < \infty$, let C_α^v be the level-curve $V_\alpha(x,y)=v$

ρ_α^v = rotation number of $f_\alpha | C_\alpha^v$

ρ_α = rotation number of the linearisation of f_α at the fixed point.





TWIST-MAP

THEOREM 5

α	0	$0<\alpha<1$	1	$1<\alpha<\infty$	∞
ρ_α^v	$\frac{1}{6}$	$\frac{1}{6}<\rho_\alpha^v<\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}<\rho_\alpha^v<\frac{1}{4}$	$\frac{1}{4}$

(Proof uses algebraic geometry)

THEOREM 6

$$\rho_\alpha^v \xrightarrow[v \to v_\alpha]{} \rho_\alpha = \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{1+\sqrt{1+4\alpha}}\right)$$

(Proof uses differential geometry)

THEOREM 7

$$\rho_\alpha^v \xrightarrow[v \to \infty]{} \frac{1}{5}$$

(Proof uses analysis)

THEOREM 8

For large v, $\rho_\alpha^v \simeq \dfrac{\ln v}{5\ln v - \ln\alpha}$

(Estimate uses analysis)

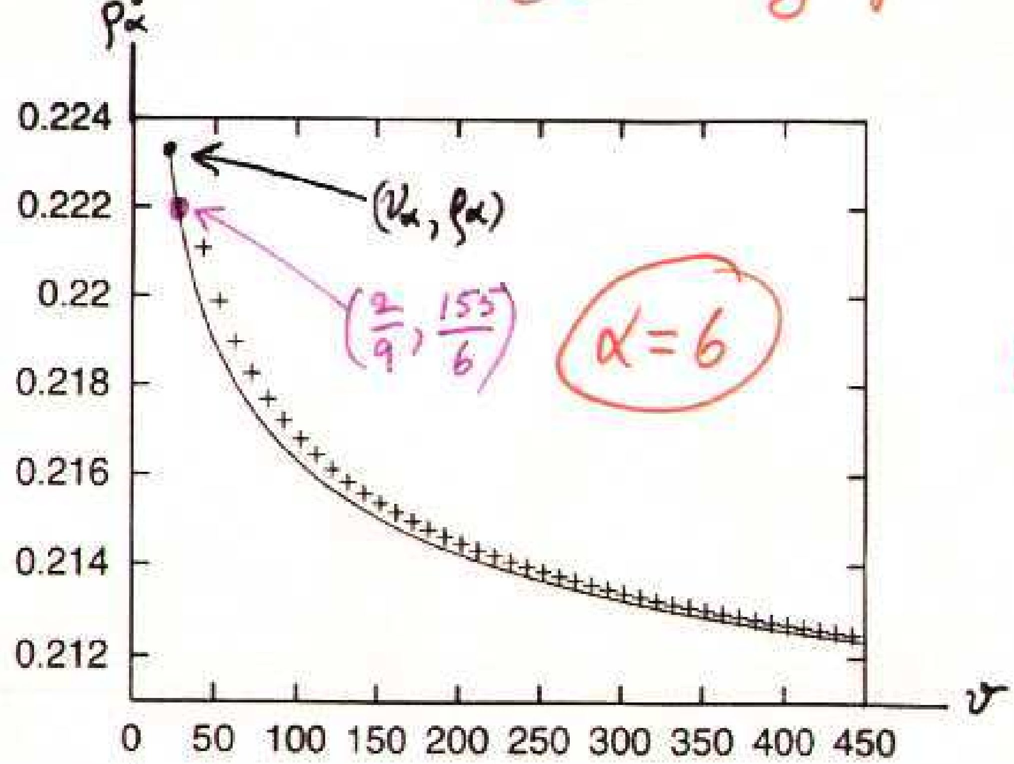
COMPUTER DRAWINGS by Colin Sparrow

Smooth line = computer estimate of ρ_α^v

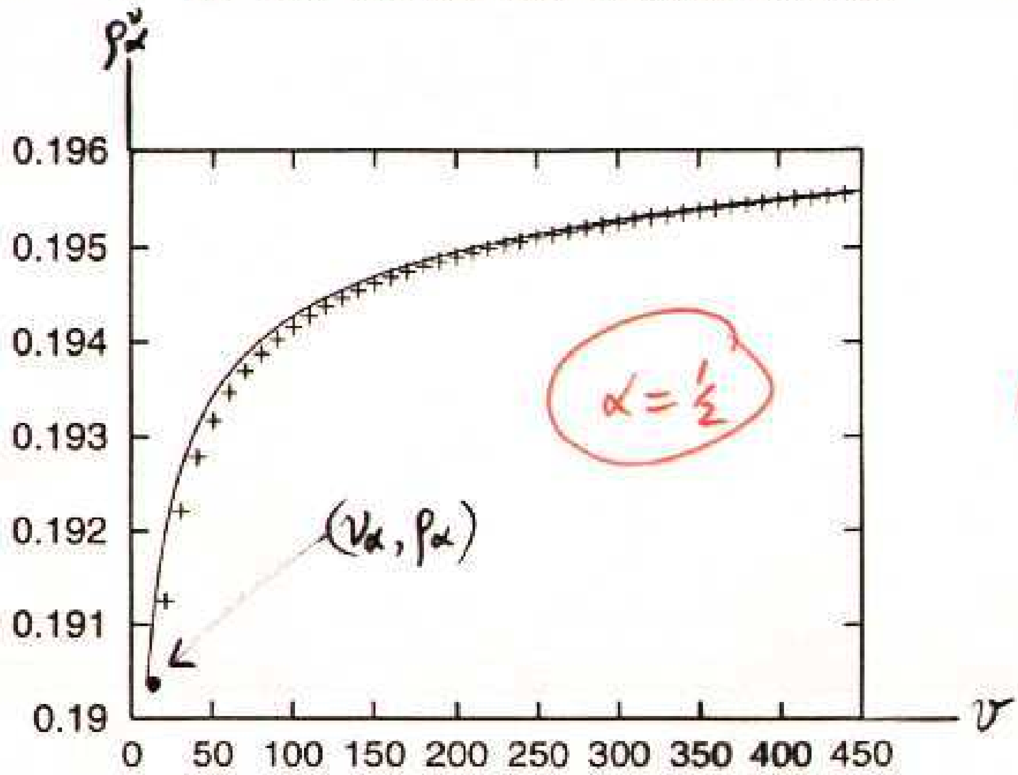
Crosses +++ = analytic estimate $\rho_\alpha^v = \dfrac{\ln v}{5\ln v - \ln \alpha}$

CONJECTURE

Graph is strictly $\begin{cases} \text{decreasing if } 1<\alpha<\infty \\ \text{increasing if } 0<\alpha<1 \end{cases}$



$1<\alpha<\infty$



$0<\alpha<1$

PERIODIC ORBITS

THEOREM 9

The set Q of periods of periodic orbits of $\{f_\alpha ; \alpha \neq 0, 1, \infty\}$ is

9, 11, 13, 14, 16, 17, 19, & all integers ≥ 21 except 42.

PROOF

$$Q = \{q \,;\, \exists \text{ coprime } p,\ \tfrac{p}{q} \neq \tfrac{1}{5},\ \tfrac{1}{6} < \tfrac{p}{q} < \tfrac{1}{4}\}$$
$$= \{q \,;\, \quad " \quad " \quad " \quad \tfrac{q}{6} < p < \tfrac{q}{4}\}$$

For each q, check the p's in this window

The ones crossed out are not coprime. ($\therefore 42 \notin Q$)

q	p	q	p	q	p	q	p	q	p
1	-	11	2	21	4,5	31	6,7	41	7,8,9,10
2	-	12	-	22	$\cancel{4}$,5	32	$\cancel{6}$, 7	42	$\cancel{8}$, $\cancel{9}$,$\cancel{10}$
3	-	13	3	23	4,5	33	$\cancel{6}$,7,8	43	8,9,10
4	-	14	3	24	5	34	$\cancel{6}$,7,$\cancel{8}$	44	$\cancel{8}$, 9,$\cancel{10}$
5	1	15	$\cancel{3}$	25	$\cancel{5}$,6	35	6,$\cancel{7}$,8	45	8, $\cancel{9}$,$\cancel{10}$, 11
6	-	16	3	26	5,$\cancel{6}$	36	7,$\cancel{8}$	46	$\cancel{8}$, 9,$\cancel{10}$, 11
7	-	17	3,4	27	5,$\cancel{6}$	37	7,8,9	47	8,9,10,11
8	-	18	$\cancel{4}$	28	5,$\cancel{6}$	38	7,$\cancel{8}$,9	48	$\cancel{9}$,$\cancel{10}$, 11
9	2	19	4	29	5,6,7	39	7,8,$\cancel{9}$	49	9,10,11,12
10	$\cancel{2}$	20	$\cancel{4}$	30	$\cancel{6}$,7	40	7,$\cancel{8}$,9	50	9,$\cancel{10}$, 11,$\cancel{12}$

For $q \geq 45$ $\exists$ prime p in the window by the prime number theorem, & that will suffice.

(Hault Brady)

18A The words "By the prime number theorem" hides a multitude of ~~hard work~~ sins, for which I'm indebted to Roger Heath-Brown.

Actually the window $\to \infty$ as $\xi \to \infty$, & the prime no. theorem says # primes in this window $\to \infty$ as $\xi \to \infty$

$\therefore$ for suff large ξ $\exists$ at least one prime in the window.

But to find what is "suff large" needs estimates for the PNT

Using Tchebychev's estimate you can fiddle around to show 100,000 is suff large. Then you have to sweat away & find suitable primes to cover the range from 45 to 100,000

Alternatively use a more sophisticated modern estimate which depends on numerical computations of zeros of Riemann Zeta

Working 100,000 down to 347 there are 8 primes to fill the gap.

ORBITS OF PERIOD 9

THEOREM 10

All orbits on C_α^v are 9-periodic $\iff \begin{cases} \alpha > \dfrac{1-2\cos\frac{4\pi}{9}}{4\cos^2\frac{4\pi}{9}} = 5.411\ldots. \\ v = \dfrac{(\alpha-1)(\alpha^2-\alpha+1)}{\alpha} \end{cases}$

EXAMPLE

Let $\alpha = 6$. Then $v = \frac{155}{6}$. Let $x_1 = \frac{25}{6}$. Then

$$S = \left\{ \frac{25}{6}, \frac{29+5\sqrt{19}}{6}, \frac{13+\sqrt{19}}{5}, \frac{6(192-31\sqrt{19})}{305}, \frac{6(47-6\sqrt{19})}{61}, \frac{6(47+6\sqrt{19})}{61}, \frac{6(192+31\sqrt{19})}{305}, \frac{13-\sqrt{19}}{5}, \frac{29-5\sqrt{19}}{6}, \frac{25}{6}, \ldots \right\}$$

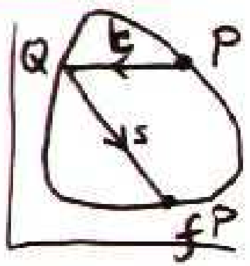
CONJECTURE

$\nexists$ rational 9-periodic sequence

SKETCH PROOF of THM 10

$f|C$ is the composition of two involutions

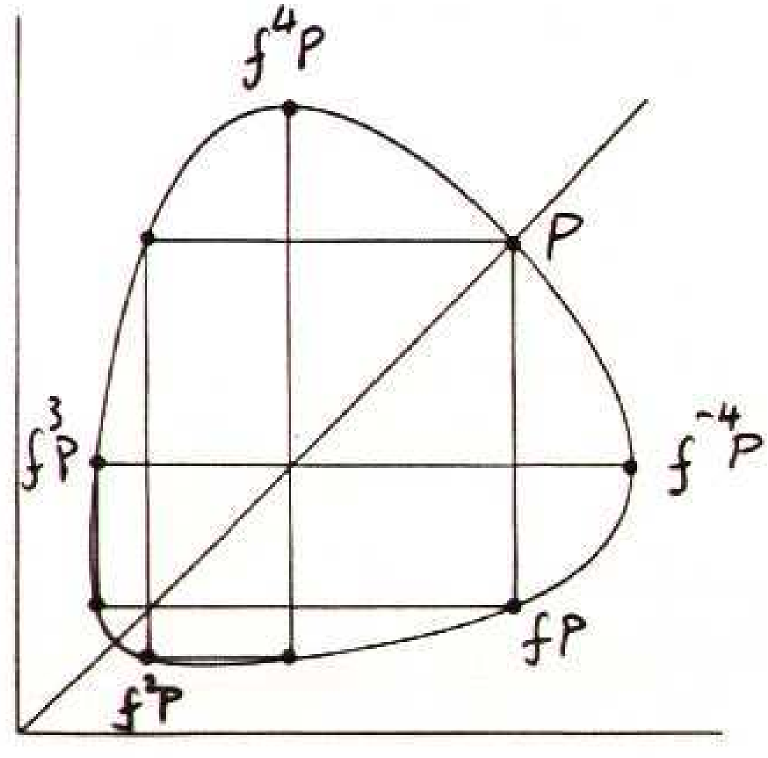
$$(x,y) \xrightarrow[\text{horizontal}]{t} \left(\frac{\alpha+y}{x}, y\right) \xrightarrow[\text{reflect in diagonal}]{s} \left(y, \frac{\alpha+y}{x}\right)$$



Start with P on the diagonal.

To get a 9-cycle the tangent at f^4P must be horizontal.

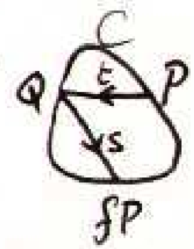
Substitute in the vanishing derivative & do lots of algebra.



PROOF OF THEOREM 3 : $f|C$ is rotation-like

Recall C is the cubic $(x+1)(y+1)(x+y+\alpha) - \nu xy = 0$ in $\mathbb{R}^2_+$.

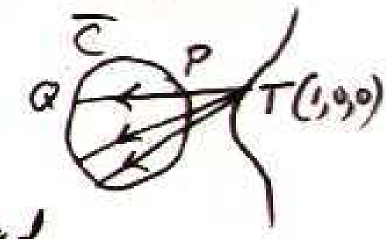
Let $\bar{C}$ be the completion of C in the complex projective plane, given by $(x+z)(y+z)(x+y+\alpha z) - \nu xyz = 0$.



Recall $f|C = st$, product of two involutions.

Similarly $\bar{f}|\bar{C} = \bar{s}\bar{t}$.

Now $\bar{s}, \bar{t}$ are induced by points $\left\{ \begin{array}{l} S = (1,-1,0) \\ T = (1,0,0). \end{array} \right.$



Since $\bar{C}$ is elliptic, it can be represented

$$\mathbb{R}^2 \xrightarrow[\text{project}]{\pi} \underset{\text{torus}}{\mathbb{R}^2/\mathbb{Z}^2} \xrightarrow[\text{diffeo}]{h} \bar{C}$$

such that if $P_1, Q_1, T_1 \in \mathbb{R}^2 \xmapsto{\quad h\pi \quad} P, Q, T \in \bar{C}$

then $P_1 + Q_1 + T_1 = 0 \implies P, Q, T$ collinear.

$\therefore$ the involution $\bar{t} : \bar{C} \to \bar{C}$ lifts to involution $t_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$$P_1 \longmapsto -P_1 - T_1$$

Therefore the composition $\bar{f}|\bar{C} = \bar{s}\bar{t}$ lifts to

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{\quad \bar{t} \quad} & \mathbb{R}^2 & \xrightarrow{\quad \bar{s} \quad} & \mathbb{R}^2 \\ P_1 & \longmapsto & -P_1 - T_1 & \longmapsto & -(-P_1 - T_1) - S_1 \\ & & & & = P_1 + (T_1 - S_1) \end{array}$$

which is translation by the vector $\rho = T_1 - S_1$.

Now C lifts to a line $L \subset \mathbb{R}^2$, parallel to ρ.

$\therefore f|C$ lifts to the translation ρ of L, which projects to the euclidean rotation r_ρ of the circle πL.

$\therefore f|C$ lifts to r_ρ. $\therefore$ $f|C$ is rotation-like

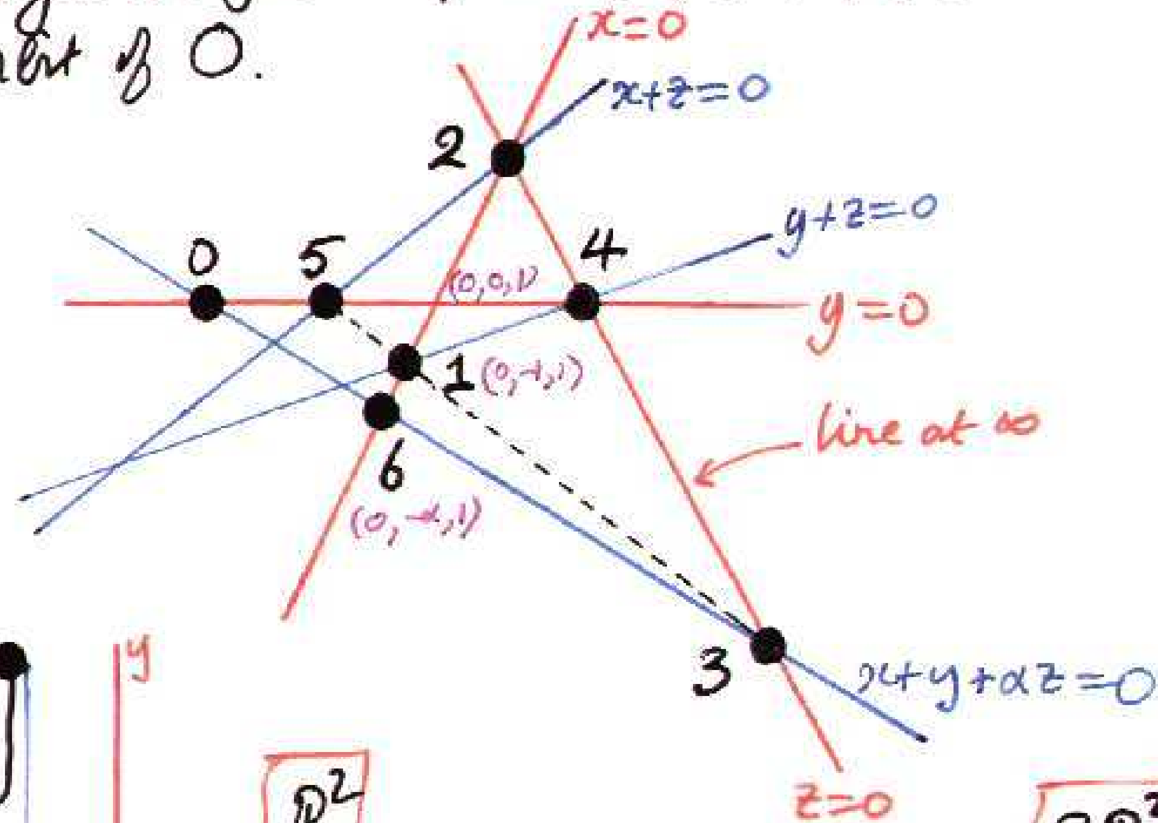
Swinnerton-Dyer

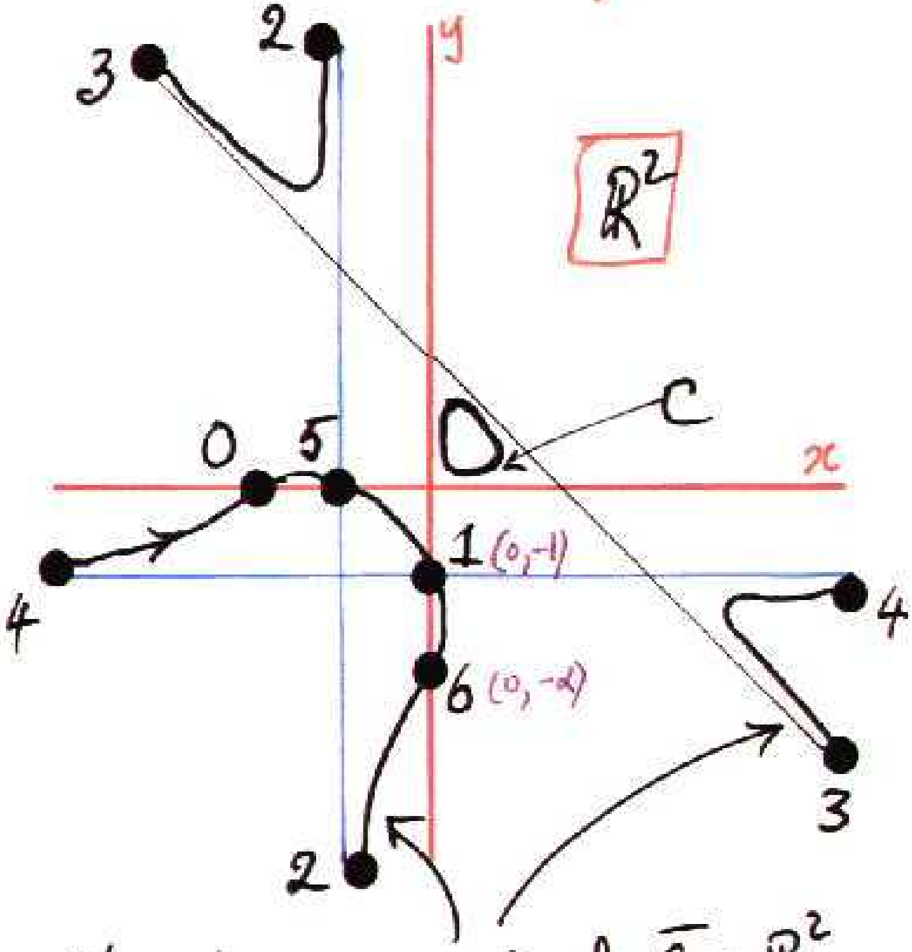
PROOF OF THEOREM 5 $\begin{cases} 0<\alpha<1 \Rightarrow \frac{1}{6}<\rho<\frac{1}{5} \\ 1<\alpha<\infty \Rightarrow \frac{1}{5}<\rho<\frac{1}{4} \end{cases}$

The pencil of cubics $(x+z)(y+z)(x+y+\alpha z) - v\,xyz = 0$ all go through the first 7 points $\{0,1,2,3,4,5,6,\cdots\}$ on the orbit of O.

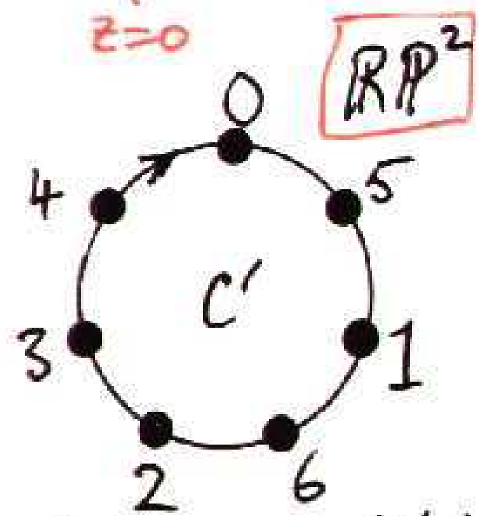
Complex projective plane.

$\boxed{\mathbb{CP}^2}$





C' = other components of $\bar{C} \cap \mathbb{R}^2$



C' has <u>same</u> rotation number ρ as C.

$4\rho < 1 < 5\rho.$

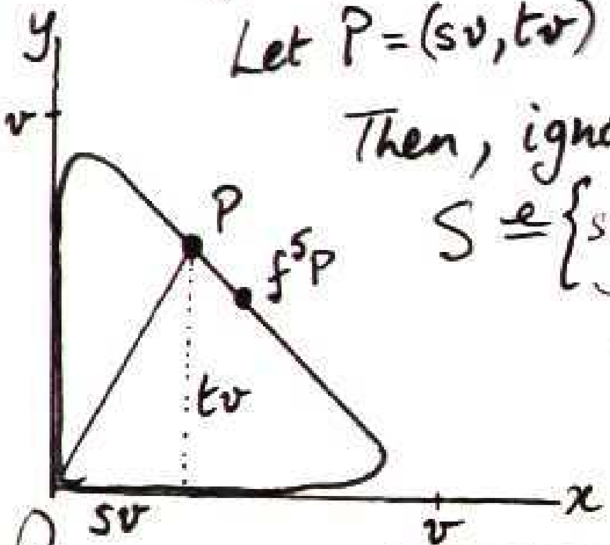
$\therefore \frac{1}{5} < \rho < \frac{1}{4}.$

PROOF OF THEOREM 8 $\quad \rho_\alpha^v \doteq \dfrac{\ln v}{5\ln v - \ln\alpha}$, if v large.

Suppose $\alpha > 1$. C is given by $\dfrac{(x+1)(y+1)(x+y+\alpha)}{xy} = v$.

For large v, $C \doteq$ triangle with hypotenuse $x+y = v$.

Let $P = (sv, tv)$ be on the hypotenuse, where $s+t = 1$.



Then, ignoring small terms,

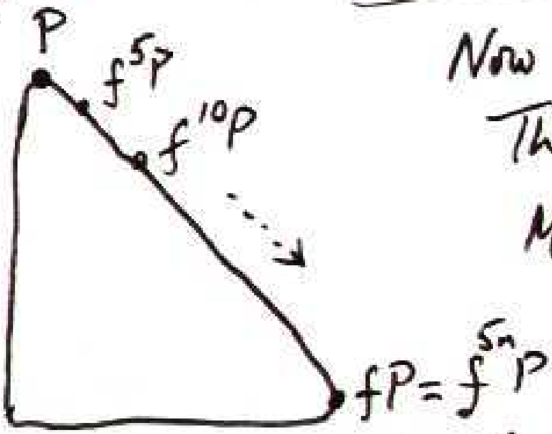
$$S \doteq \Big\{ \underbrace{sv, tv}_{P}, \frac{t}{s}, \frac{\alpha s+t}{stv}, \frac{\alpha s}{t}, \underbrace{\frac{\alpha s v}{\alpha s+t}, \frac{vt}{\alpha s+t}}_{f^5P}, \dots \Big\}$$

Slope of $OP = \dfrac{t}{s}$

Slope of $Of^5P = \dfrac{t}{\alpha s} = \dfrac{\text{slope } OP}{\alpha}$

Now let $P = (\sqrt{v}, v)$. (on $x^2 = y+\alpha$.)

Then $fP = \left(v, \dfrac{\alpha + v}{\sqrt{v}}\right) \doteq (v, \sqrt{v})$



Meanwhile $P, f^5P, f^{10}P$ marches slowly down the hypotenuse until, say,

$$fP = f^{5n}P.$$

Then $\begin{cases} \text{slope } OP = \dfrac{v}{\sqrt{v}} = \sqrt{v} \\ \text{slope } OfP = \dfrac{\sqrt{v}}{v} = \dfrac{1}{\sqrt{v}} = \dfrac{\text{slope } OP}{\alpha^n} = \dfrac{\sqrt{v}}{\alpha^n}. \end{cases}$

$\therefore \alpha^n = v. \quad \therefore n\ln\alpha = \ln v \quad \therefore n = \dfrac{\ln v}{\ln\alpha}$

During each 5 steps the orbit rotates once round C plus a bit.
After $5n$ steps the orbit reaches fP, so $5n\rho = n + \rho$

$$\therefore (5n-1)\rho = n. \quad \therefore \rho = \frac{n}{5n-1} = \frac{\ln v}{5\ln v - \ln\alpha}$$

Similarly when $\alpha < 1$

To prove $\rho_\alpha^\nu \xrightarrow[\nu \to \infty]{} \frac{1}{5}$,

require rigorous ε-δ techniques & $<$

rather than estimates $\simeq$