DUFFING'S EQUATION

Professor Sir Christopher Zeeman.

G. Duffing introduced his equation in 1918 to study electronics.

It is the simplest nonlinear forced damped oscillator.

Gradual changes in the forcing frequency can cause catastrophic jumps in amplitude & phase.

References

1. G. Duffing: Erzwungene Schwingungen bei veränderlicher Eigenfrequenz, Braunschweig, 1918
2. J.J. Stoker: Nonlinear vibrations, Interscience, 1950
3. E.C. Zeeman: Duffing's equation in brain modelling, Bull. IMA 12 (1976) 207-214.
4. These transparencies: math.utsa.edu/ecz

DUFFING'S EQUATION IS THE SIMPLEST NONLINEAR FORCED DAMPED OSCILLATOR.

simple harmonic oscillator with frequency 1

small damping

small nonlinearity

small periodic forcing term with frequency $\Omega = 1 + \varepsilon\omega$ near that of the oscillator

$$\boxed{\overbrace{\ddot{x} + x} + \varepsilon k \dot{x} + \varepsilon \alpha x^3 = \varepsilon F \cos \Omega t}$$

where x = variable

t = time

$\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$

ε is a small constant > 0

k, α, F, ω are parameters with $k, F > 0$

$\alpha, \omega \gtreqless 0.$

damping

nonlinearity

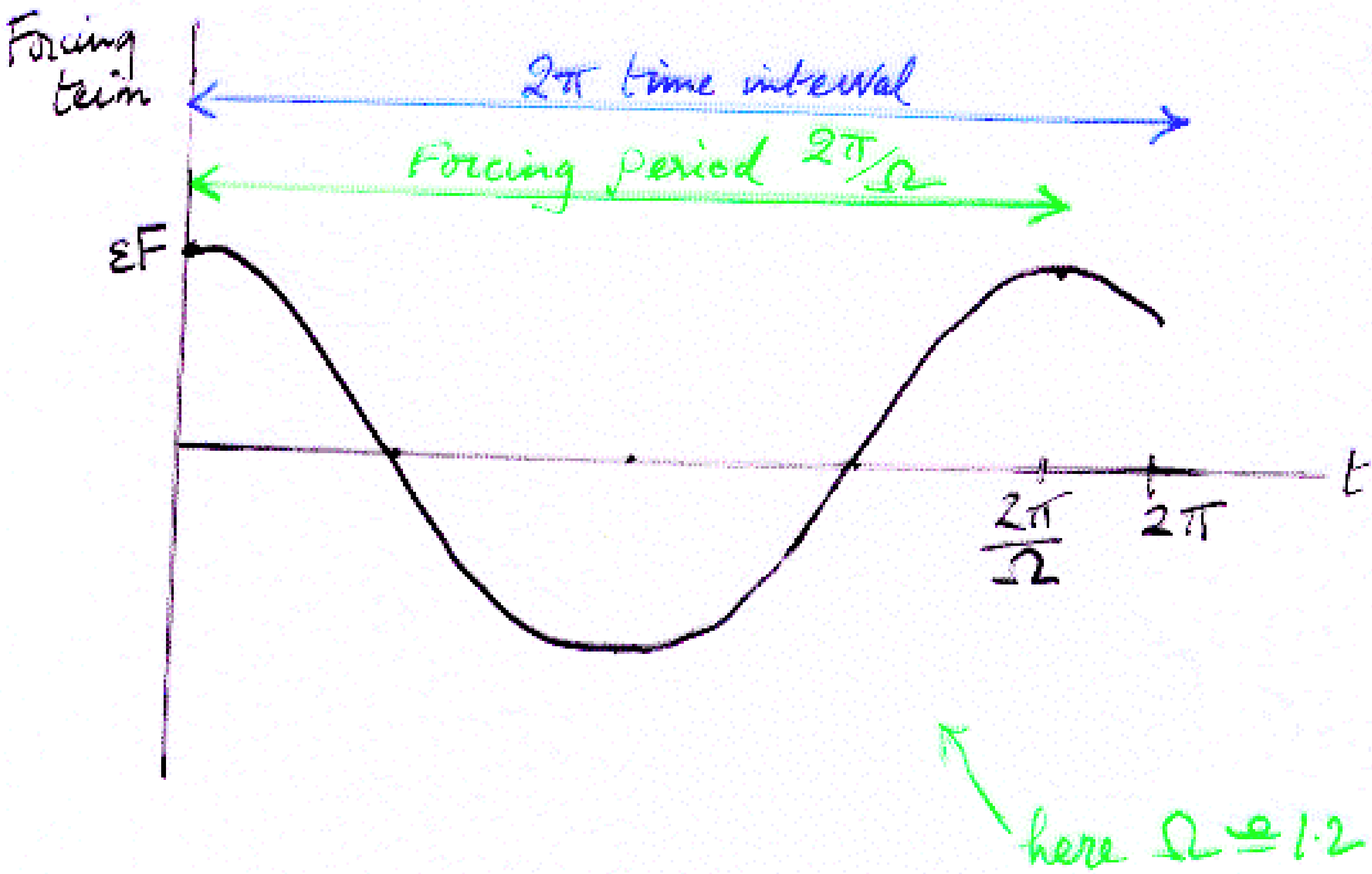
forcing amplitude

forcing frequency $\Omega = 1 + \varepsilon\omega$

FORCING TERM $= \varepsilon F \cos \Omega t$

Forcing <u>period</u> $= \dfrac{2\pi}{\Omega}$ = time between two peaks

Forcing <u>frequency</u> $= \Omega$ = number of periods in a 2π time interval.



PROGRAMME.

1. Simple harmonic oscillator
2. Damped oscillator
3. Forced oscillator
4. Forced damped oscillator.

all linear

5. Nonlinear forced damped oscillator = Duffing

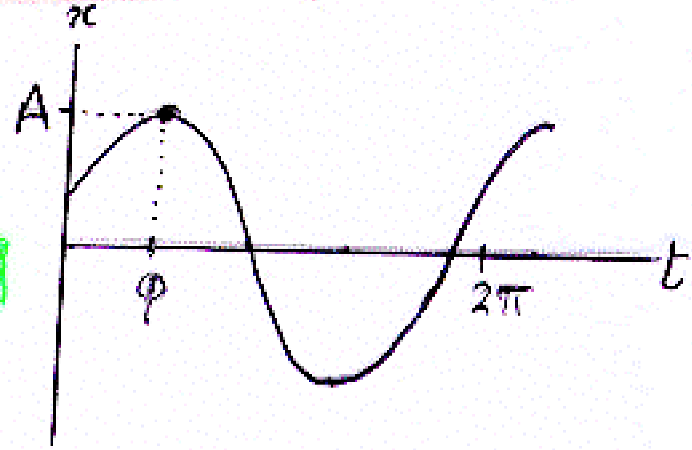
1 SIMPLE HARMONIC OSCILLATOR $\ddot{x} + x = 0$

- Solution $x = A\cos(t-\varphi)$

amplitude

frequency 1

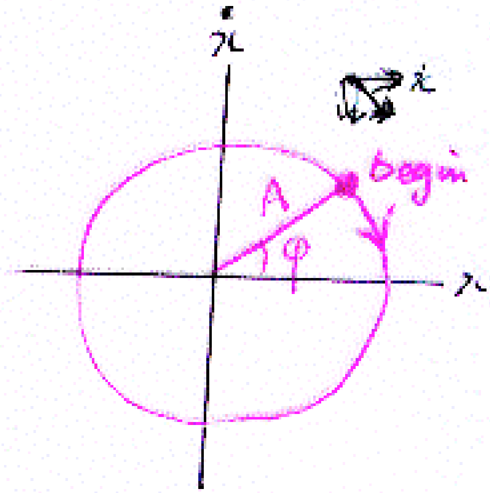
initial phase-lag



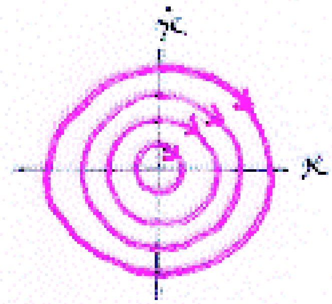
- Phase-space $\mathbb{R}^2$, with coordinates $x, \dot{x}$

$$x = A\cos(t-\varphi) = A\cos(\varphi - t)$$
$$\dot{x} = -A\sin(t-\varphi) = A\sin(\varphi - t)$$

Orbit = circle radius A, clockwise.



- Phase space foliated by concentric circles.



- Structurally unstable (∴ useless)

(phase portrait can be changed by arbitrarily small perturbations).

② DAMPED OSCILLATOR

$$\ddot{x} + \varepsilon k \dot{x} + x = 0$$

ε small > 0
$k > 0$.

- Solution: put $x = Ae^{\lambda t}$

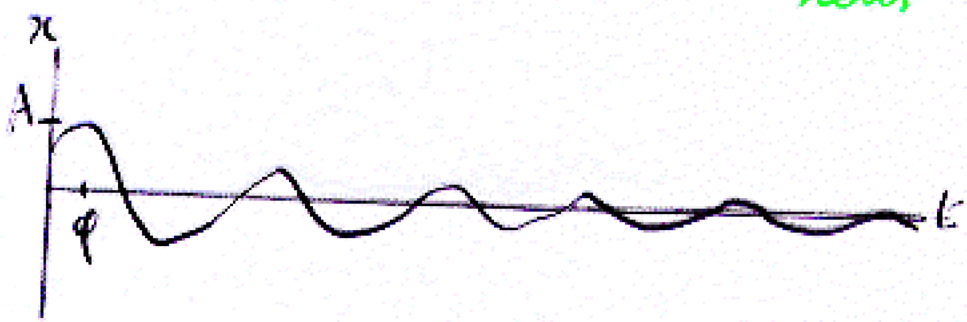
$$\therefore \lambda^2 + \varepsilon k \lambda + 1 = 0$$

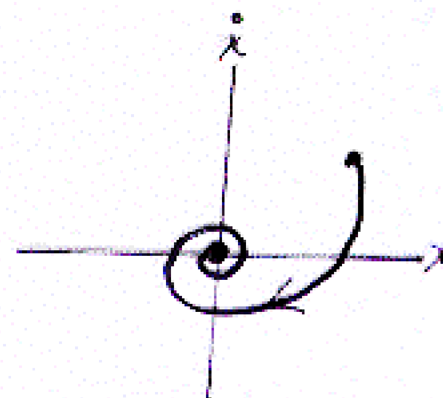
$$\therefore \lambda = \frac{-\varepsilon k \pm \sqrt{\varepsilon^2 k^2 - 4}}{2} = -\frac{\varepsilon k}{2} \pm i\psi$$

where $\psi = \sqrt{1 - \frac{\varepsilon^2 k^2}{4}} = 1 + O(\varepsilon^2)$

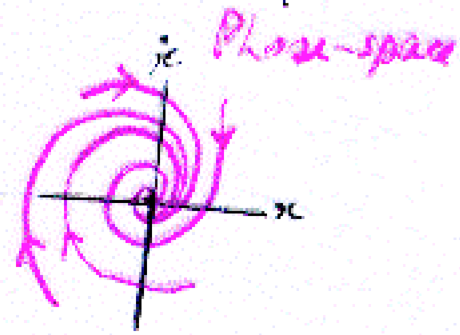
$$\therefore x = A\, e^{-\frac{\varepsilon k}{2} t} \cos(\psi t - \varphi)$$

initial amplitude — transient — frequency near 1 — initial phase-lag





- Phase portrait foliated by spirals.
  O is an attractor.



- Structurally stable ($\therefore$ useful)
  (phase portrait remains homeomorphic under sufficiently small perturbations).

③ FORCED OSCILLATOR $\ddot{x} + x = \varepsilon F \cos \Omega t$,

where ε small >0

$F > 0$

$\Omega = 1 + \varepsilon w$

$\Omega^2 = 1 + 2\varepsilon w + \varepsilon^2 w^2$

- PI (Particular Integral) = sol^n with freq Ω. [{Solution written above "Integral"]

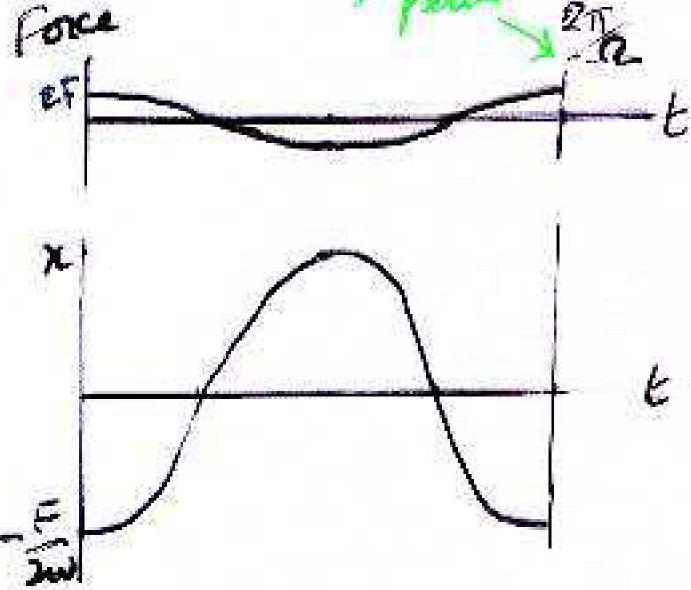
Let $x = A\cos\Omega t$. $\therefore -\Omega^2 A + A = \varepsilon F$.

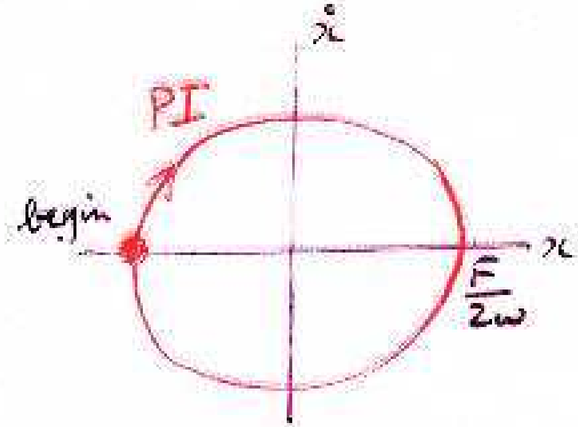
$$\therefore A = \frac{\varepsilon F}{-\Omega^2+1} = \frac{\cancel{\varepsilon} F}{-2\cancel{\varepsilon} w - \varepsilon^2 w^2} = -\frac{F}{2w} + O(\varepsilon)$$

$\therefore$ to order ε, $x = -\frac{F}{2w}\cos\Omega t = \frac{F}{2w}\cos(\Omega t - \pi)$

large amplitude

phase lag of π $(w>0)$





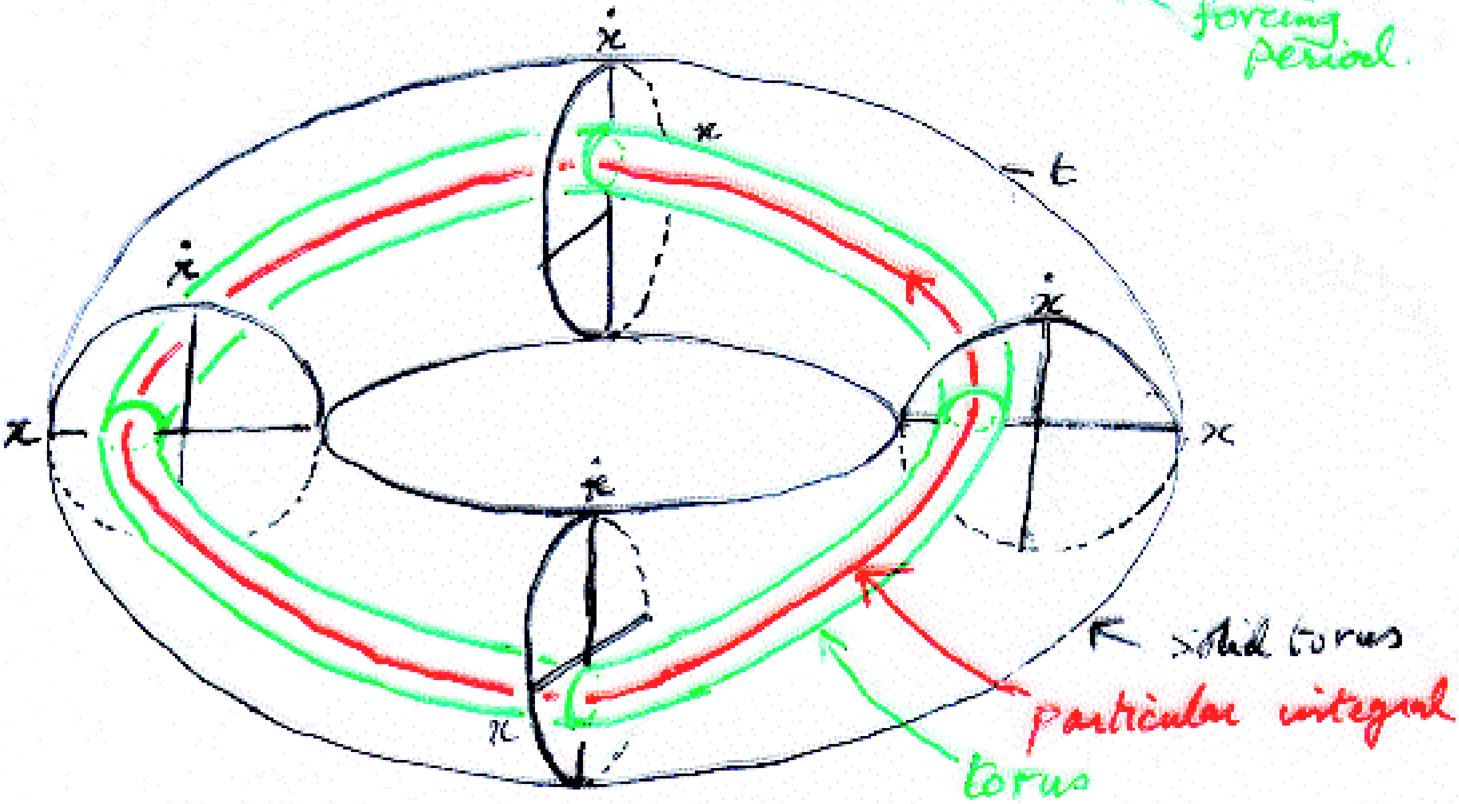
- General Solution $= \frac{F}{2w}\cos(\Omega t - \pi) + B\cos(t - \varphi)$

PI (particular integral)

Homogeneous solution

③ FORCED OSCILLATOR (cont.) $\ddot{x} + x = \varepsilon F \cos \Omega t$.

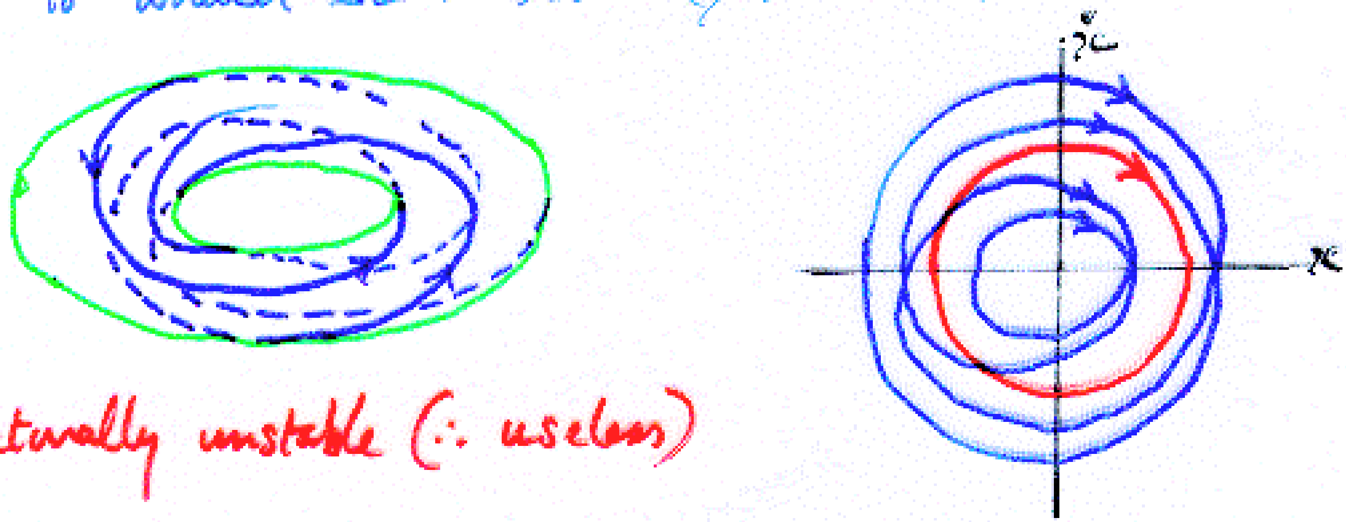
- Phase-space = solid torus $\mathbb{R}^2 \times S^1$, with coords $x, \dot{x}$, $t \bmod \frac{2\pi}{\Omega}$ ← forcing period.



The particular integral goes once round the solid torus.

The phase-space is foliated by tori enclosing the P.I.

Each torus is foliated by cycles/spirals according as to whether Ω is rational/irrational.



- Structurally unstable ($\therefore$ useless)

④ FORCED DAMPED OSCILLATOR $\ddot{x} + \varepsilon k \dot{x} + x = \varepsilon F \cos \Omega t$

where ε small > 0
$k, F > 0$
$\Omega = 1 + \varepsilon w$
$\therefore \Omega^2 = 1 + 2\varepsilon w + \varepsilon^2 w^2$

- PI (particular integral) $x = \underset{\text{amplitude}}{A} \cos(\underset{\text{freq}}{\Omega} t - \underset{\text{phase}}{\varphi})$

Let $\theta = \Omega t - \varphi$.
$\therefore \Omega t = \theta + \varphi$

$$\therefore x = A \cos\theta$$
$$\dot{x} = -\Omega A \sin\theta$$
$$\ddot{x} = -\Omega^2 A \cos\theta$$

Subst. in the equation:

$$-\Omega^2 A \cos\theta - \varepsilon k \Omega A \sin\theta + A\cos\theta = \varepsilon F(\cos\theta\cos\varphi - \sin\theta\sin\varphi)$$

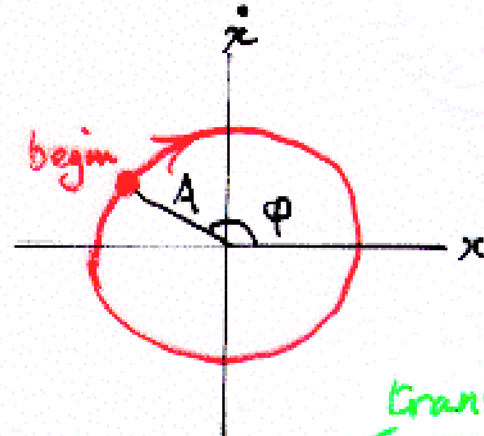
Compare coeffs of $\begin{cases} \sin\theta : & -\varepsilon k \Omega A = -\varepsilon F \sin\varphi \\ \cos\theta : & -\Omega^2 A + A = \varepsilon F \cos\varphi \end{cases}$

$$\therefore F\sin\varphi = k\Omega A = kA + O(\varepsilon)$$
$$F\cos\varphi = \frac{1-\Omega^2}{\varepsilon} A = -2wA + O(\varepsilon)$$

$$\therefore F^2 = k^2A^2 + 4w^2A^2 + O(\varepsilon)$$

$\therefore$ amplitude $\quad A = \dfrac{F}{\sqrt{k^2 + 4w^2}} + O(\varepsilon)$

phase $\quad \cot\varphi = -{}^{2w}/_{k} + O(\varepsilon)$



- General solution $\quad x = \underbrace{\dfrac{F}{\sqrt{k^2+4w^2}} \cos(\Omega t - \varphi)}_{\text{PI (particular integral)}} + \underbrace{\overset{\text{transient}}{B e^{-\frac{\varepsilon k t}{2}}} \cos(\mu t - \varphi')}_{\text{homogenous solution}}$

- Phase-space $= R^2 \times S' =$ solid torus.
  PI is an attractor: all other orbits spiral towards it.

- Structurally stable ($\therefore$ useful).

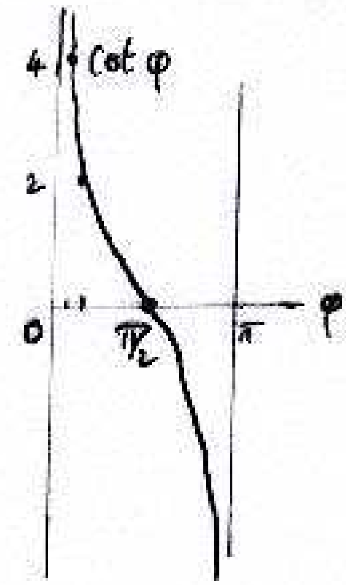
(4) FORCED DAMPED OSCILLATOR (continued)

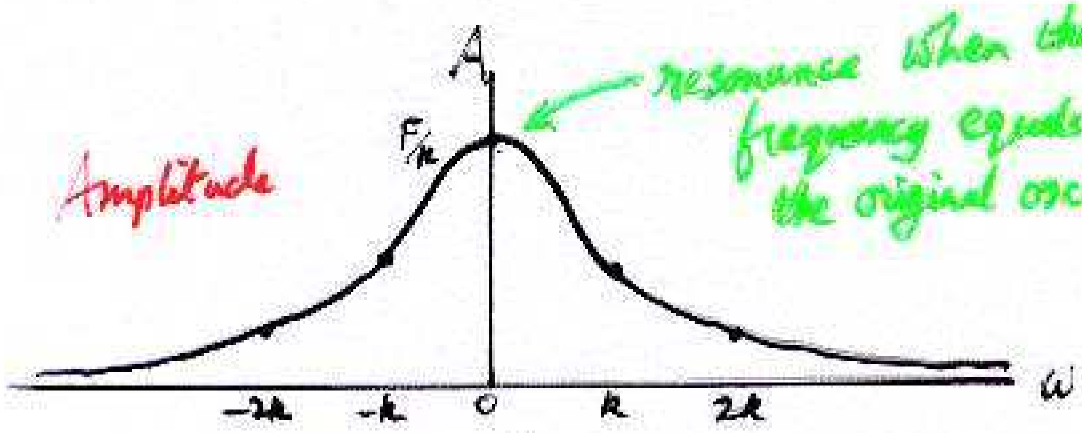
Forcing frequency $\Omega = 1 + \varepsilon\omega$.

Take ω as a parameter.

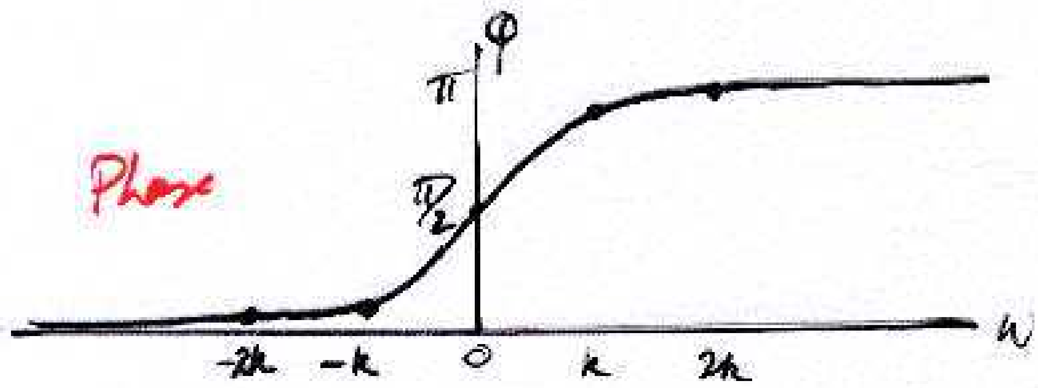
Amplitude $A = \dfrac{F}{\sqrt{k^2 + 4\omega^2}}$, Phase $\cot\varphi = -\dfrac{2\omega}{k}$

ω	A	$\cot\varphi$	φ
∞	0	$-\infty$	π
$2k$	$F/k\sqrt{17}$	-4	$\frac{13}{14}\pi$
k	$F/k\sqrt{5}$	-2	$\frac{6}{7}\pi$
0	F/k	0	$\pi/2$
$-k$	$F/k\sqrt{5}$	2	$\pi/7$
$-2k$	$F/k\sqrt{17}$	4	$\pi/14$
$-\infty$	0	∞	0





resonance when the forcing frequency equals that of the original oscillator, $\Omega = 1$, $\omega = 0$



(5) DUFFING'S EQUATION

$$\ddot{x} + \varepsilon k\dot{x} + x + \varepsilon\alpha x^3 = \varepsilon F\cos\Omega t$$

where ε small >0

$k, F > 0$

$\alpha \gtrless 0$

$\Omega = 1 + \varepsilon\omega$, $\omega \gtrless 0$

$\Omega^2 = 1 + 2\varepsilon\omega + \varepsilon^2\omega^2$

- PI (particular integral)

Let $\theta = \Omega t - \varphi$ $\quad\therefore\quad \Omega t = \theta + \varphi$.

Put $x = A\cos\theta + \varepsilon B\cos 3\theta$ ← small harmonic

$\therefore \dot{x} = -\Omega A\sin\theta - 3\Omega\varepsilon B\sin 3\theta$

$\ddot{x} = -\Omega^2 A\cos\theta - 9\Omega^2\varepsilon B\cos 3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\therefore \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

ignore ε^2.

$\therefore \ddot{x} + x = (-\Omega^2+1)A\cos\theta + (-9\Omega^2+1)\varepsilon B\cos 3\theta = -2\varepsilon\omega A\cos\theta - 8\varepsilon B\cos 3\theta$

$\varepsilon k\dot{x} = -\varepsilon kA\sin\theta$

$\varepsilon\alpha x^3 = \varepsilon\alpha A^3\cos^3\theta = \varepsilon\alpha A^3\left(\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\right)$

$\varepsilon F\cos\Omega t = \varepsilon F(\cos\theta\cos\varphi - \sin\theta\sin\varphi)$

Subst. in the equation:

$$-2\cancel{\varepsilon}\omega A\cos\theta - 8\cancel{\varepsilon}B\cos 3\theta - \cancel{\varepsilon}kA\sin\theta + \cancel{\varepsilon}\alpha A^3\left(\tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta\right) = \cancel{\varepsilon}F(\cos\theta\cos\varphi - \sin\theta\sin\varphi)$$

Compare coeffs of
$$\begin{cases} \cos\theta: & -2\omega A + \frac{3}{4}\alpha A^3 = F\cos\varphi \\ \sin\theta: & -kA = -F\sin\varphi \\ \cos 3\theta: & -8B + \frac{\alpha A^3}{4} = 0 \end{cases}$$

$\therefore B = \frac{\alpha A^3}{32}$ ← amplitude of small harmonic FORGET

$\therefore F\sin\varphi = kA$

$F\cos\varphi = A\left(\frac{3}{4}\alpha A^2 - 2\omega\right)$

Duffing Amplitude relation $\quad F^2 = k^2A^2 + A^2\left(\frac{3}{4}\alpha A^2 - 2\omega\right)^2$

Duffing Phase relation $\quad \cot\varphi = \dfrac{\frac{3}{4}\alpha A^2 - 2\omega}{k}$

Take ω as a parameter.

DUFFING AMPLITUDE RELATION

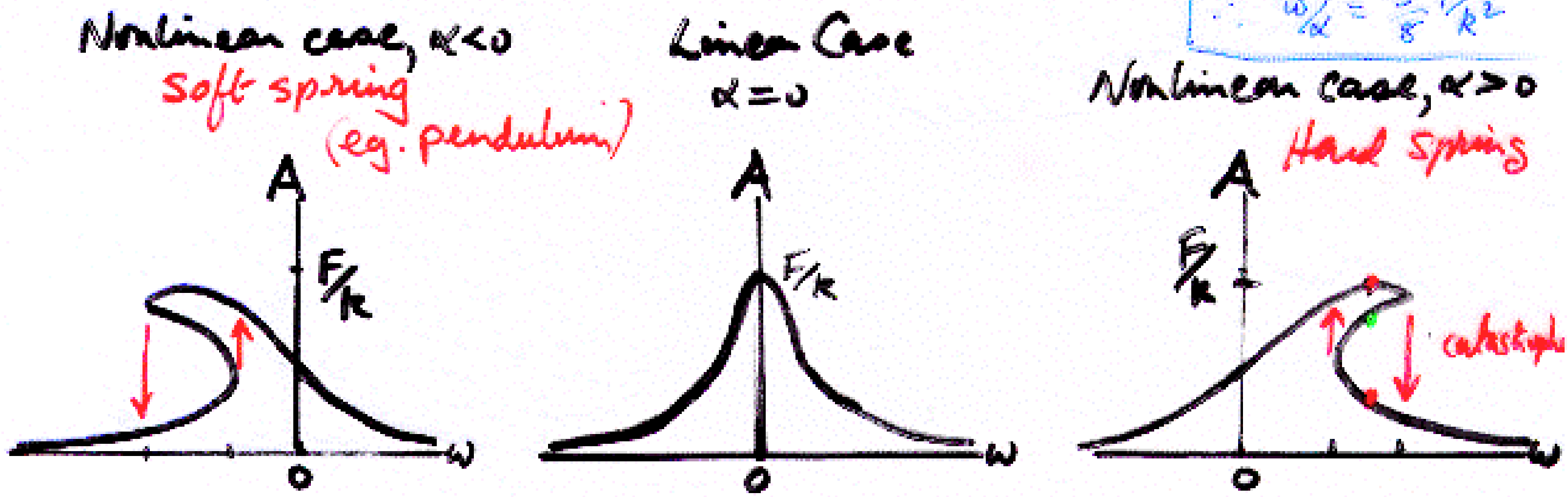
$$F^2 = k^2A^2 + A^2\left(\tfrac{3}{4}\alpha A^2 - 2\omega\right)^2$$

Maximum amplitude given by $dA/d\omega = 0$.

$\therefore -4A^2\left(\tfrac{3}{4}\alpha A^2 - 2\omega\right) = 0$

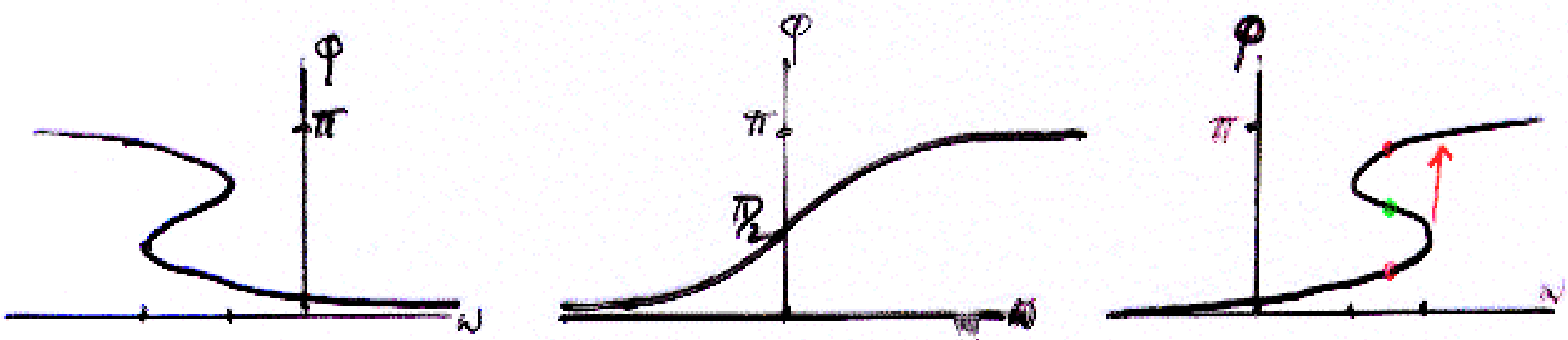
$\therefore A = F/k$

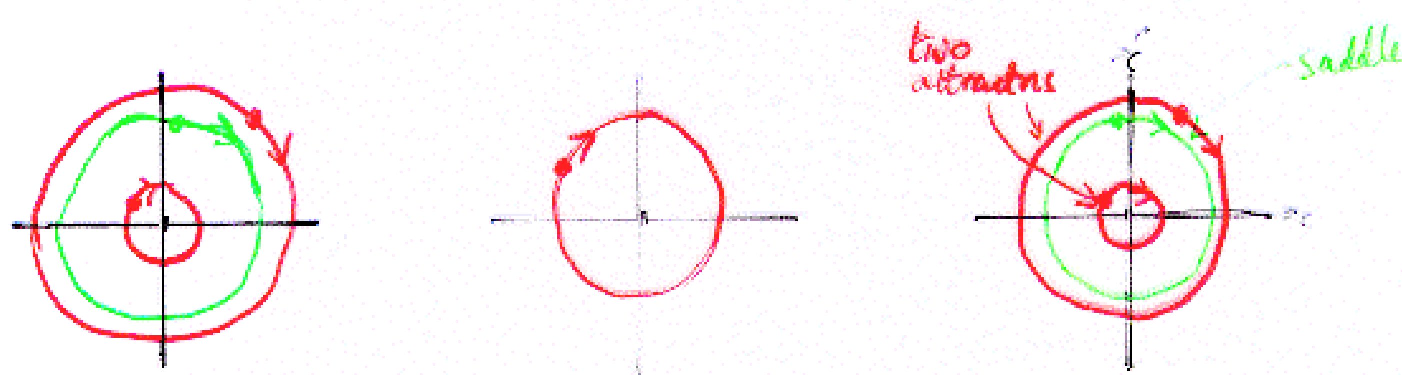
$\therefore \omega/\alpha = \tfrac{3}{8}\, F^2/k^2$

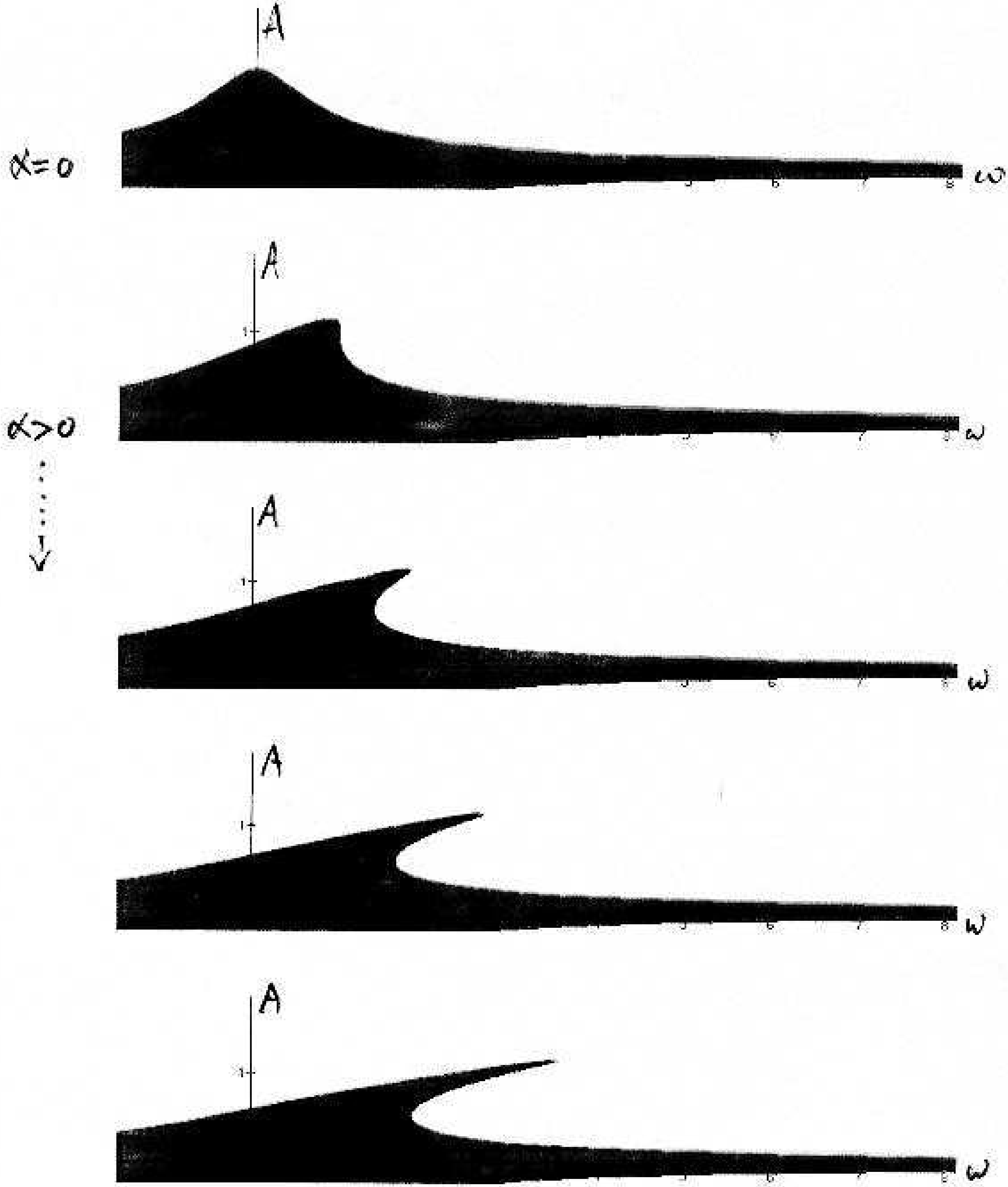


DUFFING PHASE RELATION

$$\cot\varphi = \frac{\tfrac{3}{4}\alpha A^2 - 2\omega}{k}$$



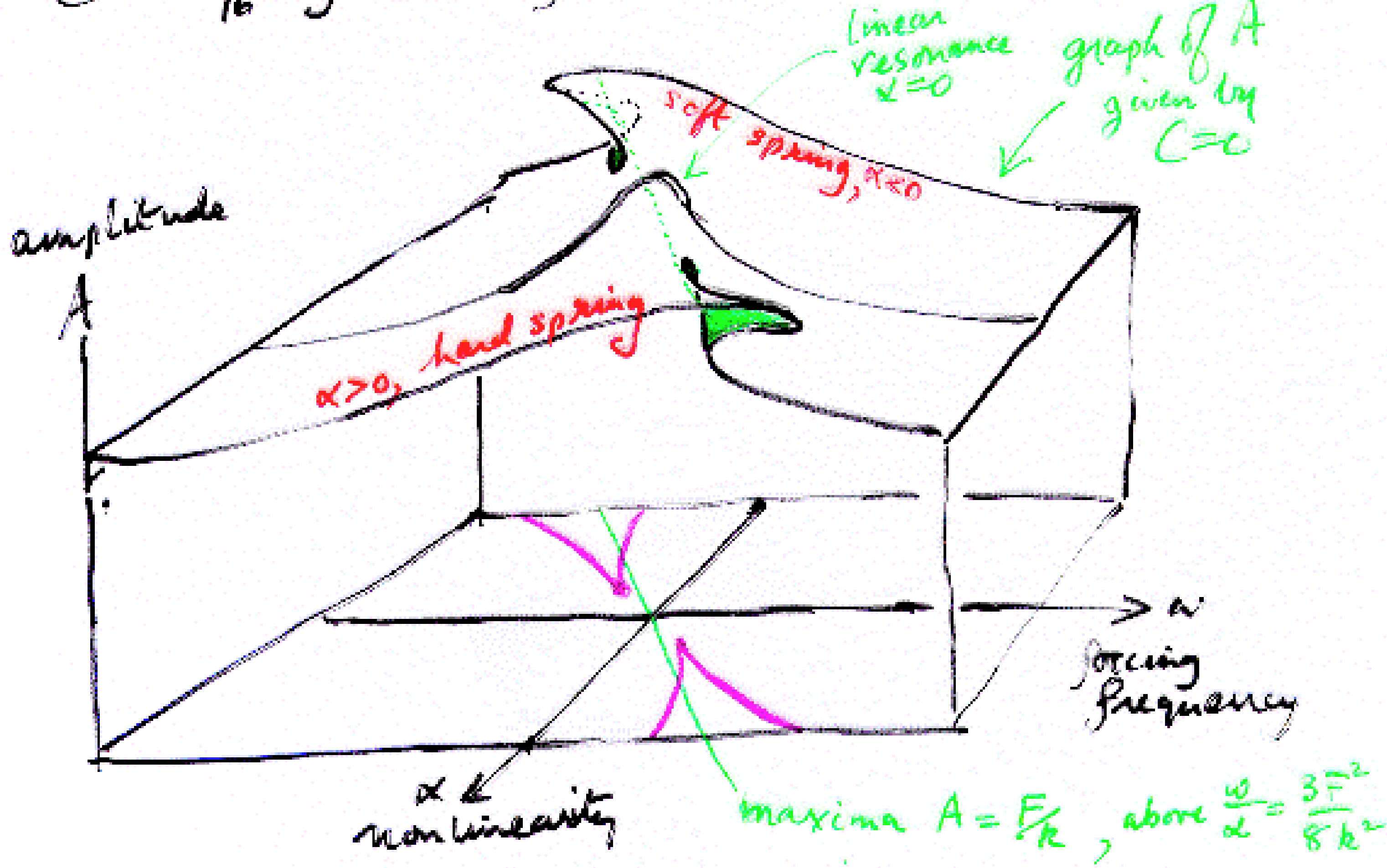



A
α=0
5
6
7
8
ω
α>0
1

GRAPH OF AMPLITUDE A OVER PARAMETERS $\begin{cases} \omega & \text{FORCING FREQUENCY} \\ \alpha & \text{NONLINEARITY} \end{cases}$

Putting $A^2 = y$, we can write the Duffing amplitude relation as a cubic in y:

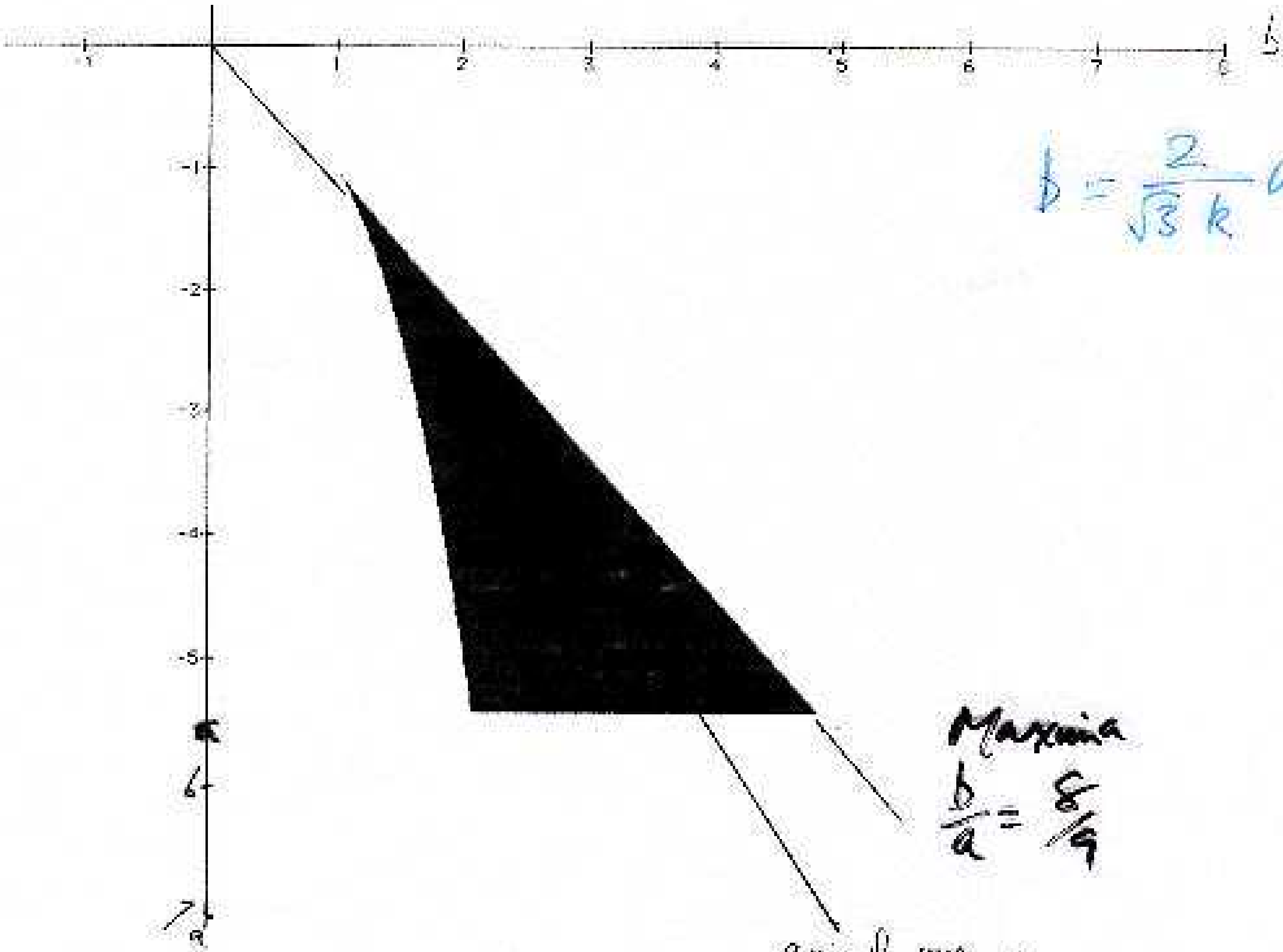
$$C \equiv \tfrac{9}{16}\alpha^2 y^3 - 3\alpha\omega y^2 + (k^2 + 4\omega^2)y - F^2 = 0.$$



Bimodal inside cusps given by $C = C' = 0$

Cusp points given by $C = C' = C'' = 0$:

$$\alpha = \pm\frac{32k^3}{9\sqrt{3}F^2}, \quad \omega = \pm\frac{\sqrt{3}k}{2}, \quad A = \frac{\sqrt{3}F}{2k}$$



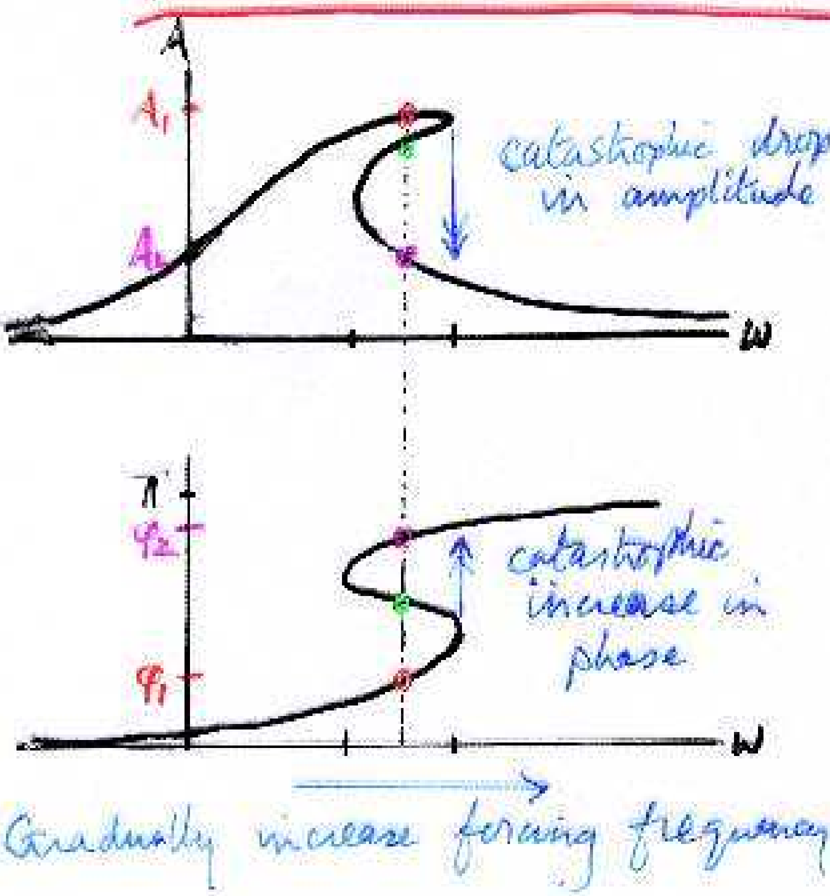
$$b = \frac{2}{\sqrt{3}\,k}\,\omega$$

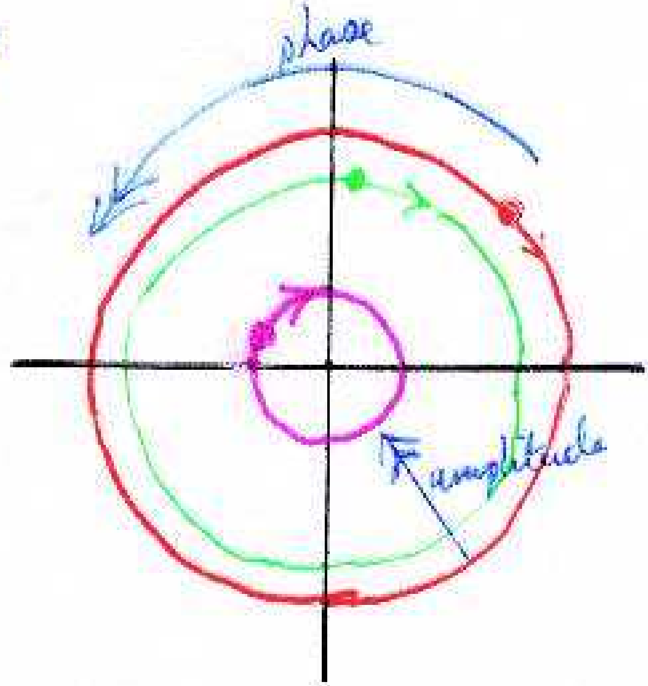
$$\frac{b}{a} = \frac{8}{9}$$

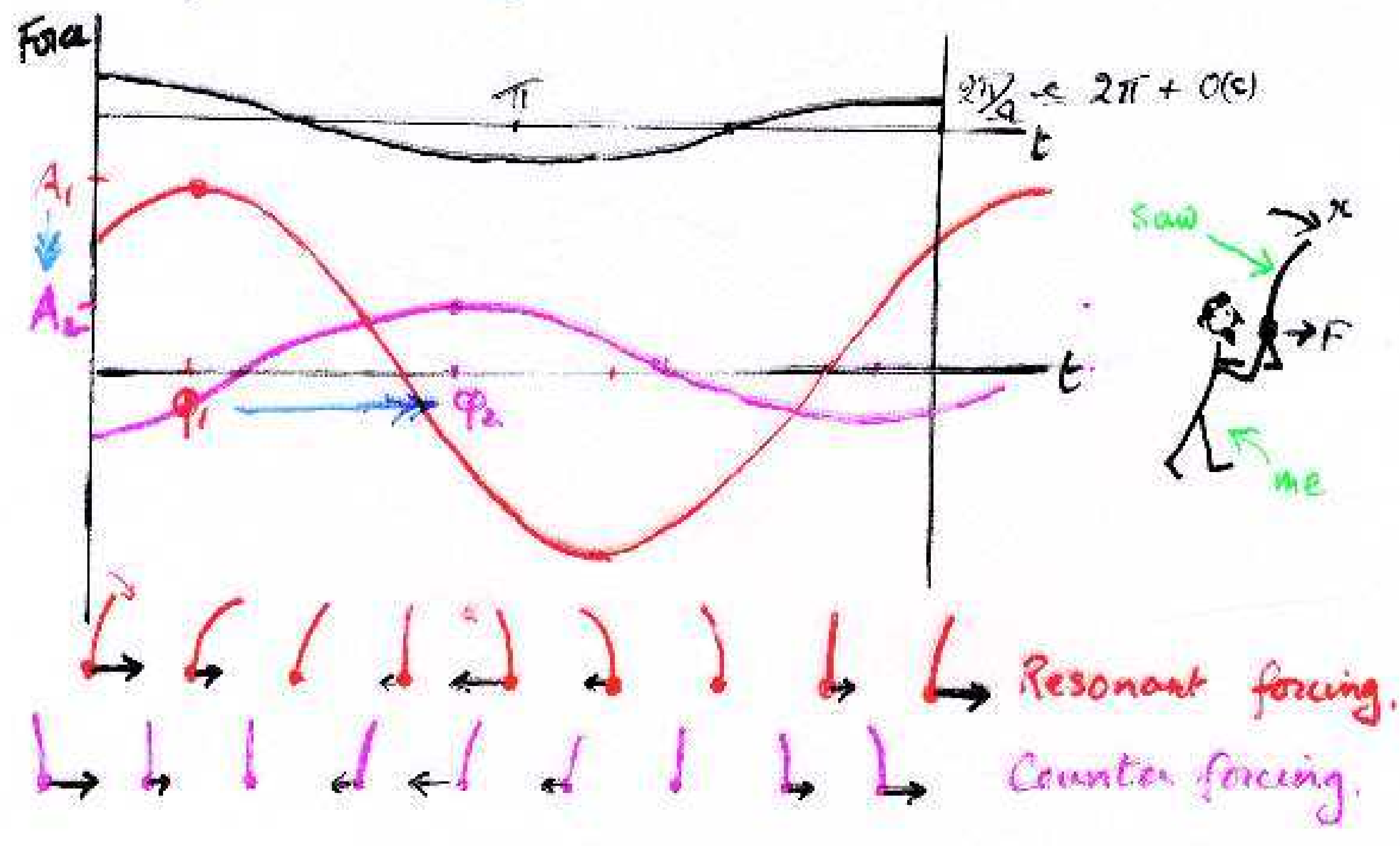
$$\frac{b-1}{a-1} = \frac{2}{3}$$

$$a = \frac{9\sqrt{3}\,F^2}{32\,k^3}\,\alpha$$

HARD SPRING EXPERIMENT ($\alpha > 0$)







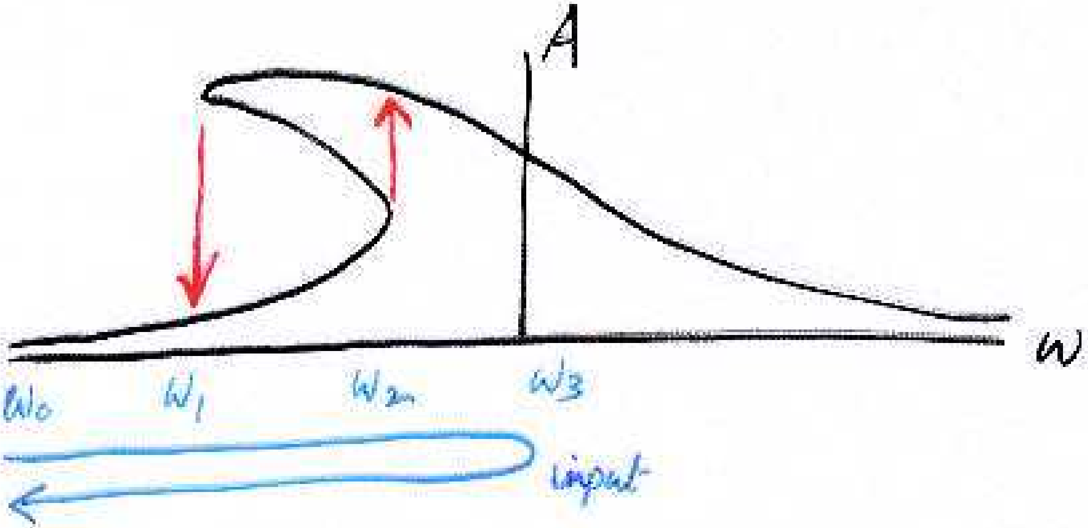
POSSIBLE APPLICATIONS TO BRAIN MODELLING.

1. Sensory inputs
2. Memory recall
3. Switches of mood
4. Anorexia/Bulimia
5. Manic/depression
6. Circadian rhythm & cortisol dysfunction
7. Regulation.

SENSORY INPUTS

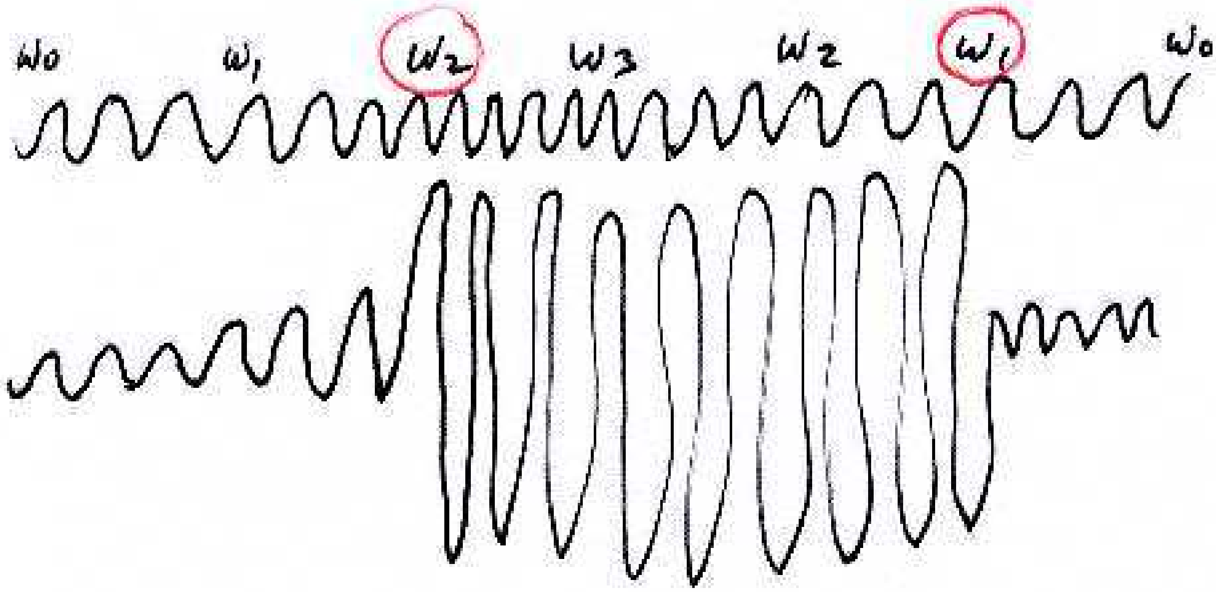
Sensory input ↑ amplitude (eg. ↑ light intensity) $\xRightarrow{\text{Sensory mechanism}}$ ↑ frequency of neuronal firing in optic nerve $\xRightarrow{\text{Duffing}}$ ↑ amplitude of response in optic cortex

Use SOFT SPRING model



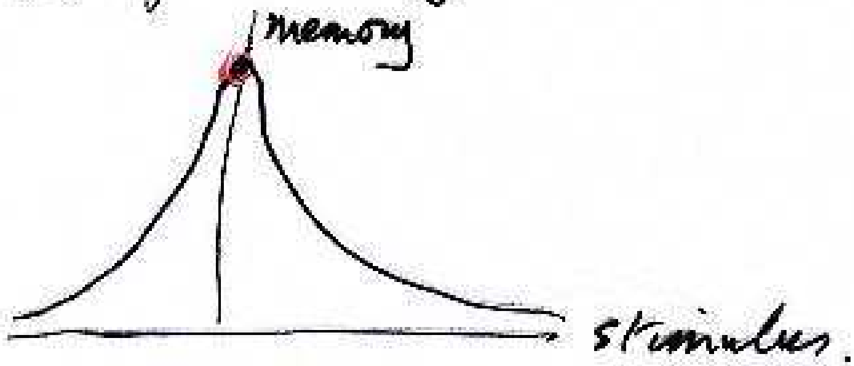
Input = Forcing term

Output = response



MEMORY RECALL

(1) <u>Laying down</u> of a memory trace modelled by resonance

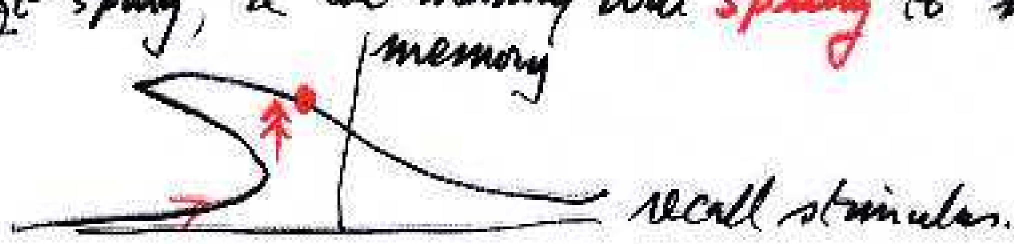


(2) <u>Recall of a familiar memory</u>.

If the memory, or closely related thoughts, have been stimulated recently, facilitating the neural networks, thereby creating a <u>hard</u> spring, then the memory flows to mind

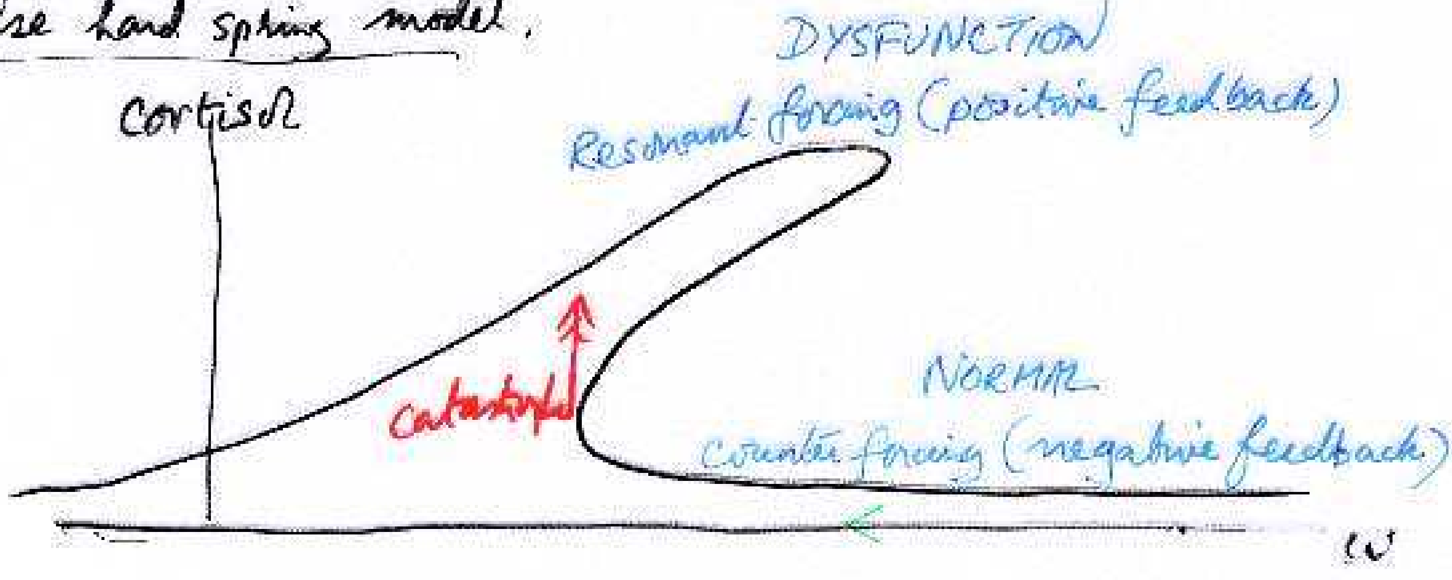
(3) <u>Recall of an unfamiliar memory</u>

If the memory has lain dormant, decay will inhibit the neural networks, thereby creating a <u>soft</u> spring, & the memory will spring to mind



CIRCADIAN RHYTHM & CORTISOL DYSFUNCTION

Use hard spring model.



Suppose

1. There is an increase in the internal circadian rhythm (the body's 24-hour clock)

2. Then there is a relative decrease in the external circadian rhythm (day & night)

3. This can cause a catastrophic switch from counter-forcing (negative feedback) to resonant-forcing (positive feedback) in the production of cortisol.

4. This causes exaggerated cortisol levels at night (sleeplessness) & lack of cortisol by day (lethargy), leading to depression. (cf. Cushing's Syndrome)