COMPLEX NUMBERS

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THE FUNDAMENTAL THEOREM OF ALGEBRA

E.C. Zeeman.

Gresham Professor of Geometry

REPRESENT ALGEBRA BY GEOMETRY

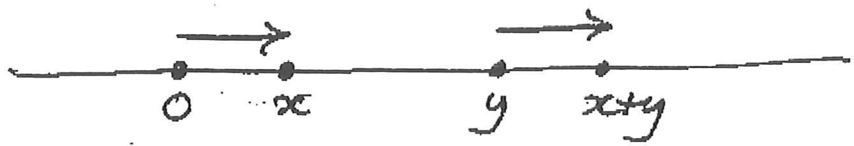
Algebra	Geometric representation
① Number x	point
② Add $x+$	translate
③ Minus $-$	rotate
④ Multiply $x\times$	expand & rotate

REAL NUMBERS

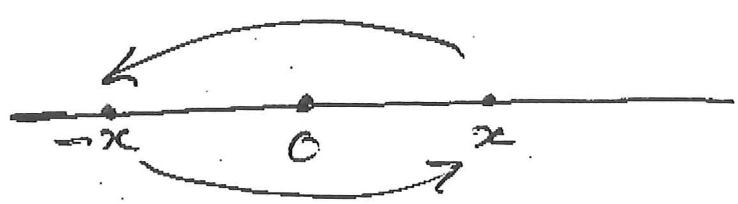
① A real number x is a point on the line

② Add. Draw an arrow from 0 to x.

$x+$ means translate parallel to this arrow



③ Minus means rotate through $180°$ about 0



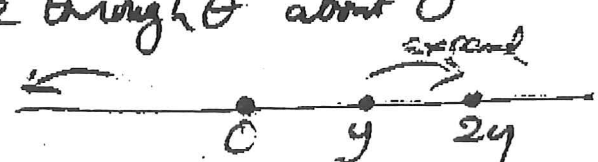
Therefore two minuses make a plus.

④ Multiply. Write x in polar coordinates

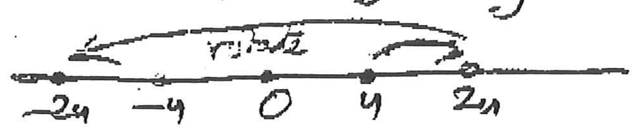
$$x = \langle r, \theta \rangle, \text{ where } \begin{cases} r \geq 0 \\ \theta = \begin{cases} 0 & \text{if } x \text{ positive} \\ 180° & \text{if } x \text{ negative} \end{cases} \end{cases}$$

$x\times$ means $\begin{cases} \text{expand (or contract) by } r \text{ about } 0 \\ \text{rotate through } \theta \text{ about } 0 \end{cases}$

Example $x = 2$.

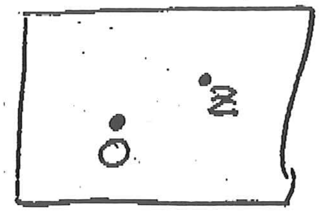


Example $x = -2$



COMPLEX NUMBERS

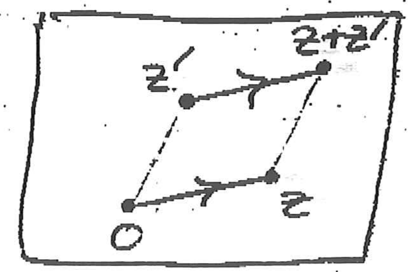
① A complex number z is a point in the plane.



② **Add.** Draw an arrow from O to z.

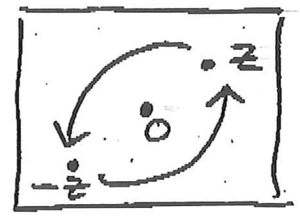
$z+$ means translate parallel to this arrow.

Therefore $z+z'$ is given by the parallelogram rule.



③ **Minus** means rotate through 180° about O

Therefore two minuses make a plus.



④ **Multiply.** Write z in polar coordinates

$$z = \langle r; \theta \rangle \quad \text{where} \begin{cases} r \geq 0 \\ 0 \leq \theta < 360^\circ \end{cases}$$

$$z\times \text{ means} \begin{cases} \text{expand (or contract) by } r \text{ about } O \\ \text{rotate through } \theta \text{ about } O \end{cases}$$

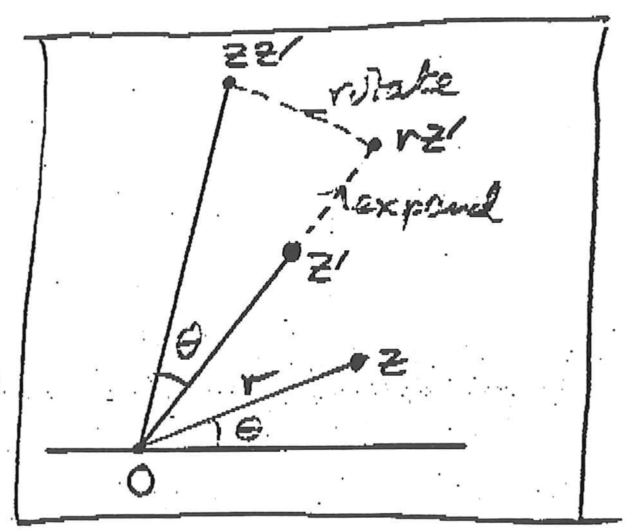
Therefore $z' = \langle r'; \theta' \rangle$

$$\Big\downarrow \text{ expand by } r$$

$$r z' = \langle r r', \theta' \rangle$$

$$\Big\downarrow \text{ rotate through } \theta$$

$$z z' = \langle r r'; \theta + \theta' \rangle$$



Therefore zz' given by $\begin{cases} \text{multiplying radii} \\ \text{adding angles} \end{cases}$

In particular $z^2 = \langle r^2; 2\theta \rangle$

PROPERTIES OF THE COMPLEX NUMBERS

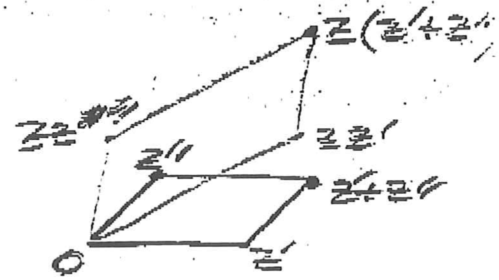
① They obey the algebraical laws:

Commutative $z + z' = z' + z$, $zz' = z'z$

Associative $(z+z')+z'' = z+(z'+z'')$ $(zz')z'' = z(z'z'')$

Distributive $z(z'+z'') = zz'+zz''$

Proof. If you expand and rotate a parallelogram it remains a parallelogram.



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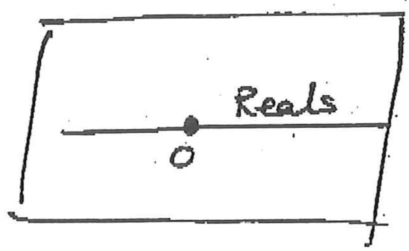
② Real numbers are contained in the complex numbers



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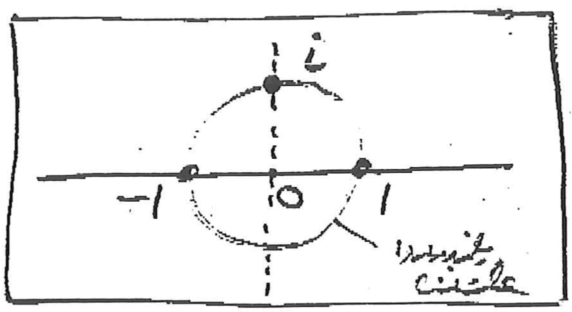
③ Define $i = \langle 1, 90^\circ \rangle$

Therefore $i^2 = \langle 1, 180^\circ \rangle = -1$

Therefore $i = \sqrt{-1}$



Multiply by i means rotate through 90°

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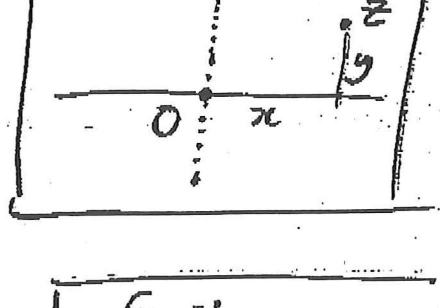
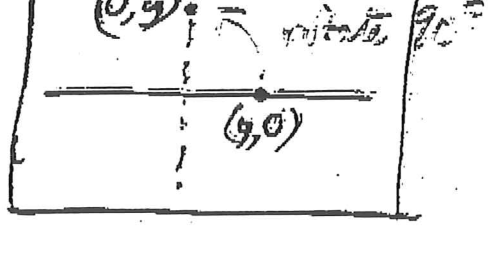
④ Cartesian coordinates.

$$
\begin{aligned}
z &= (x, y) \\
&= (x,0) + (0,y) \\
&= (x,0) + i(y,0) \\
&= x + iy
\end{aligned}
$$





HISTORY OF NUMBERS = HISTORY OF SOLVING EQUATION

TYPICAL EQUATION	SOLUTION	NEED
$x+2=5$	$x=3$	Positive integers
$x+5=2$	$x=-3$	Negative integers
$2x=1$	$x=\frac{1}{2}$	Rational numbers
$x^2=2$	$x=\sqrt{2}$	Real numbers
$x^2=-1$	$x=\sqrt{-1}$	Complex numbers.
?		?

FUNDAMENTAL THEOREM OF ALGEBRA (Gauss)

Every equation can be solved in the complex numbers

Example 1

We can solve square roots in the complex numbers

if $z=\langle r,\theta\rangle$ then $\sqrt{z}=\pm\left\langle\sqrt{r},\frac{\theta}{2}\right\rangle$

Example 2

We can solve quadratic equations

$$az^2+bz+c=0, \quad a\neq 0$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

There are two meanings to the word "solve."

① Solve directly means write the solution as a formula in the coefficients of the equation, using $\pm$, $\times$, $\div$, $\sqrt{\ }$, $\sqrt[3]{\ }$, etc.

② Solve indirectly means prove that a solution exists (but without giving a formula).

Niccolo TARTAGLIA (1500–1557) solved the cubic directly.
Ludovico FERRARI (1522–1565) solved the quartic directly.
Geronimo CARDANO (1501–1576) published these direct solutions in 1545.

For 250 years mathematicians struggled to solve the quintic equation. Eventually:

Evariste GALOIS (1811–1832) proved in 1832 that it is impossible to solve the quintic directly.

Meanwhile:

Carl Friedrich GAUSS (1777–1855) proved in 1799 that all equations can be solved indirectly (in other words the fundamental theorem of algebra).

Nowadays computers give approximate solutions with any desired degree of accuracy desired degree of accuracy

Proof

Given $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$,

where $n \geq 1$, the a_i's are complex nos.

Claim there is a z such $f(z) = 0$

If $a_0 = 0$ its easy — put $z = 0$.

$\therefore$ assume $a_0 \neq 0$

Suppose not.

i.e. suppose there is no z such that $f(z) = 0$

The we shall show a contradiction.

Let $\mathbb{C}$ denote complex plane.

Then f gives a map $f: \mathbb{C} \longrightarrow \mathbb{C}$

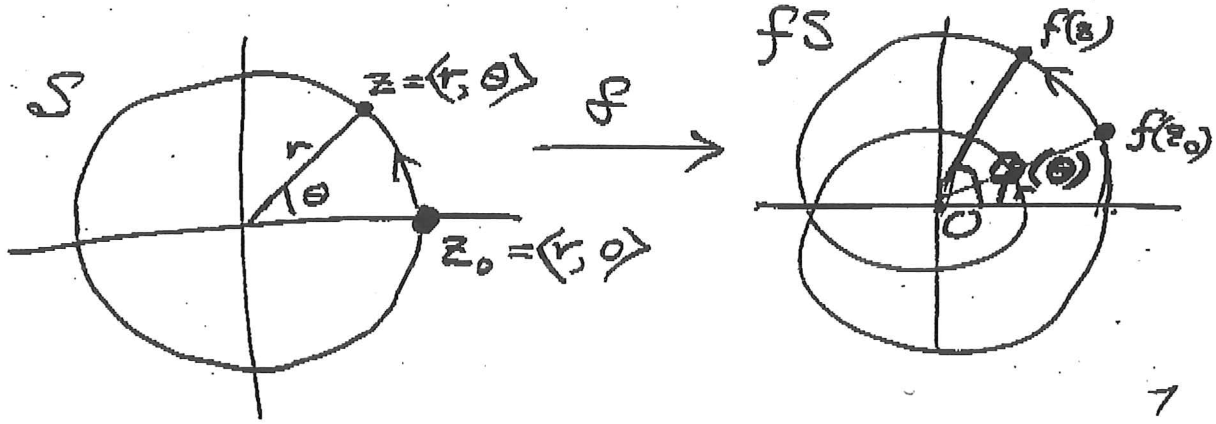
$z \longmapsto f(z)$.

(never hits 0)

The map is continuous because if we vary z a little, then $z^2, z^3, \ldots$ vary a little & so $f(z)$ varies a little.

For each $r > 0$ we shall construct a winding number W_r, as follows.

Let S be the circle radius r, centre O.



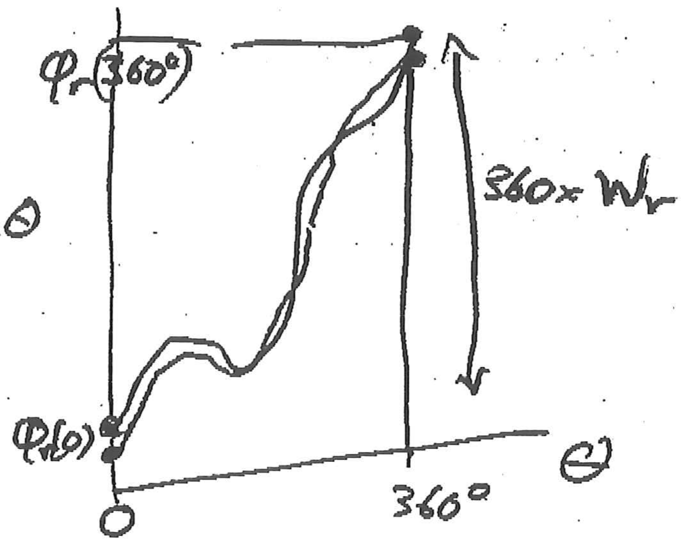
What does wind round

definition W_r = number of times $f S$ winds round O

What does "wind around" mean in the

$\varphi_r(\theta)$ = angle of $f(z)$, where $r = \langle r, \theta \rangle$

Draw the graph of φ_r



$0 \leq \varphi_r(0) < 360$

φ_r depends continuously on θ

Define $W_r = \dfrac{\varphi_r(360) - \varphi_r(0)}{360}$

<u>Lemma 1</u> <u>W_r is constant</u>

<u>Proof</u> φ_r varies continuously with r.

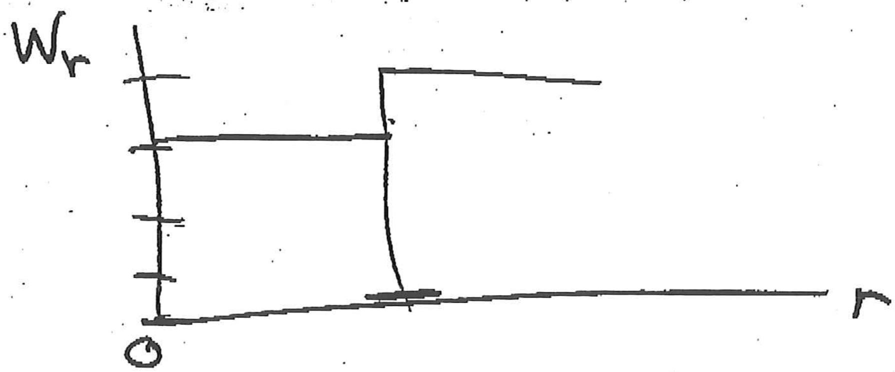
If r' is sufficiently near r then $\varphi_{r'}$ differs from φ_r by $< 180°$

$\therefore$ $360 \times W_{r'}$ differs from $360 \times W_r$ by $< 180°$

$\therefore$ $W_{r'}$ differs from W_r by $< \frac{1}{2}$

But they are both integers.

$\therefore$ $W_{r'} = W_r$.

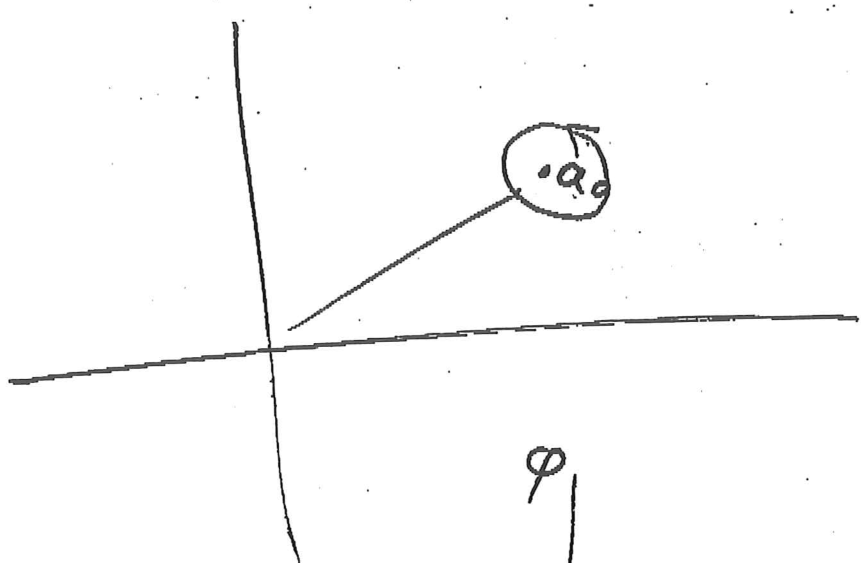


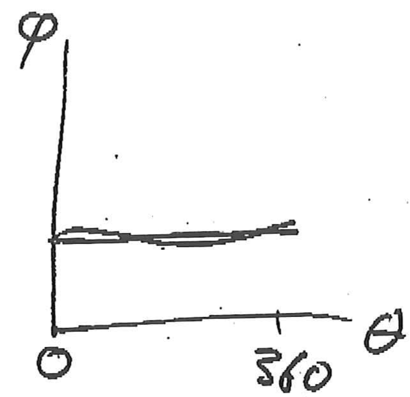
Lemma 2 For small r, $W_r = 0$

Proof $f(z) = a_0 + [a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n]$

small very small very very small

$= a_0 + \text{small}$

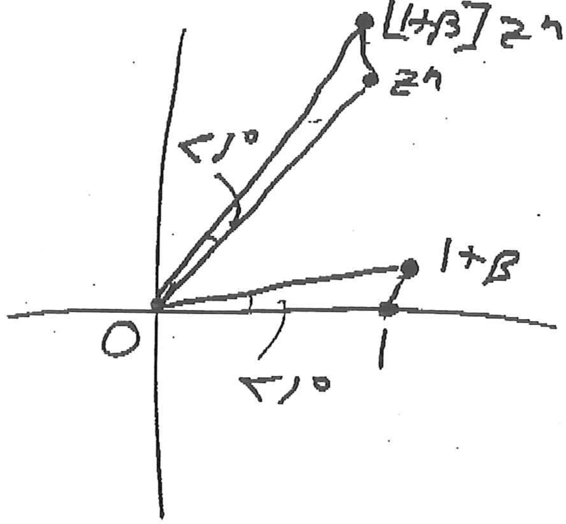




Lemma 3 For large r, $W_r = n$ $(n \geq 1)$

Proof $f(z) = \left[1 + \underbrace{\frac{a_{n-1}}{z}}_{\text{small}} + \cdots \frac{a_1}{z^{n-1}} + \underbrace{\frac{a_0}{z^n}}_{\text{very very small}} \right] z^n$

$= [1+\beta] z^n$, where β small

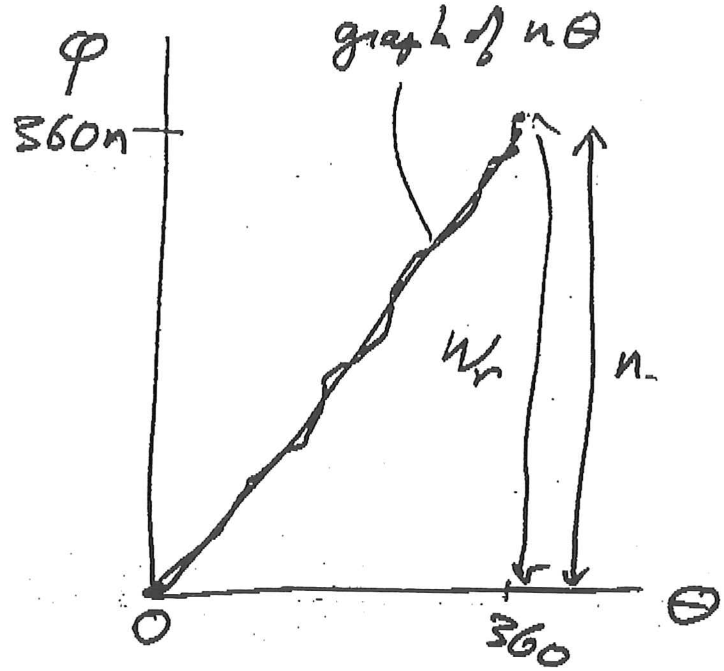


$z = \langle r, \theta \rangle$

$z^n = \langle r^n, n\theta \rangle$

angle of $z^n = n\theta$

angle of $f(z) = n\theta + (<1°)$



CONTRADICTION