THE 1993 JOHANN BERNOULLI LECTURE
GRONINGEN

By E.C. Zeeman.

CONTROVERSY IN MATHEMATICS:

On the ideas of Daniel Bernoulli & René Thom.

Nieuw Archief voor Wiskunde 11 (1993) 257–282.

THE FOUR TYPES OF APPLIED MATHEMATICS

		THINGS	
		DISCRETE	CONTINUOUS
BEHAVIOUR	DISCRETE	Dice Symmetry DISCRETE BOX Finite probability Finite groups Combinatorics	Music: harmony Light: wave/particles Discontinuities PANDORA'S BOX Fourier series Quantum theory Catastrophe theory
	CONTINUOUS	Planets Populations TIME BOX Ordinary differential equations	Waves CONTINUOUS BOX Partial differential equations

MUSIC & THE VIBRATING STRING CONTROVERSY.

500BC? PYTHAGORAS

(17) Benedetti, Beekman, Mersenne, Descartes, Huygens, Galileo

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1700 SAUVEUR : harmonics (experiment)

1713 TAYLOR : frequency, sine curve

1722–44 BACH : Das Wohltemperirtes Clavier
The 48 Preludes and Fugues

1727 Johann BERNOULLI :
(letter to his son Daniel in St. Petersburg).

1733–1740 Daniel BERNOULLI 1700–1782 : harmonics (mathematics); superposition of harmonics

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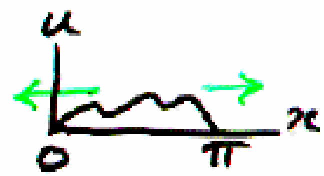
STARTS CONTROVERSY

1747 D'ALEMBERT : wave equation $u_{tt} = c^2 u_{xx}$
1717 – 1783
initially $u = f$, $u_t = 0$



solution $u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$
where f is odd and 2π-periodic

1748 EULER : non smooth curves
1707–1783

1753 Daniel BERNOULLI : $u(t,x) = \sum_{1}^{\infty} a_n \sin nx \cos nct$
where $\sum a_n \sin nx = f(x)$, initially.

RESOLVES IT

1807 FOURIER : calculated the Fourier coefficients of f.
1768–1830

1829 DIRICHLET : proved that the Fourier series converges to f.
1805–1859

STRING OF LENGTH π

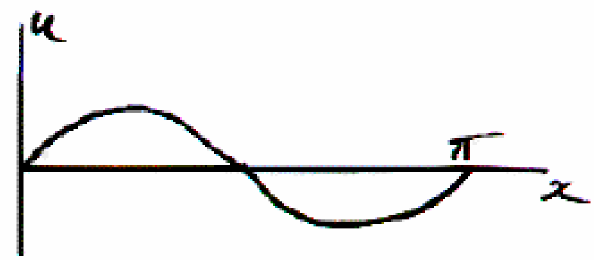
$$c = \text{speed of wave} = \sqrt{\frac{\text{tension}}{\text{mass per unit length}}}$$

The first harmonic is the fundamental:

$$u = \sin x \cos ct$$

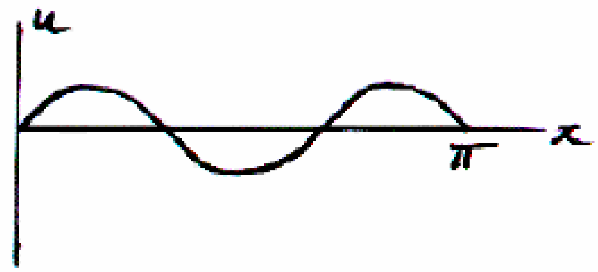
The second harmonic is the octave above:

$$u = \sin 2x \cos 2ct$$



The third harmonic is the fifth above that:

$$u = \sin 3x \cos 3ct$$



And so on . . .

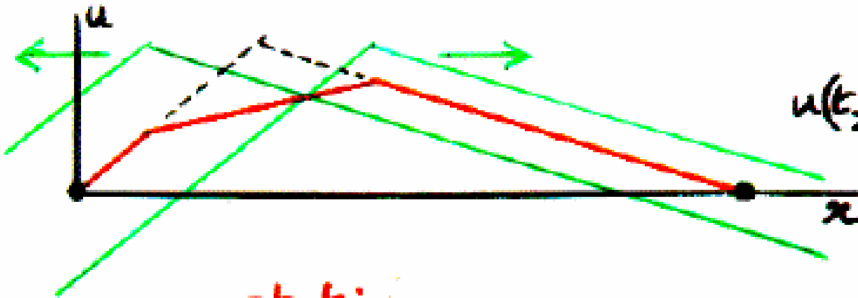
1740: DANIEL BERNOULLI

Similarly, a taut musical string can produce its isochronous tremblings in many ways and even according to theory infinitely many, . . . and moreover in each mode it emits a higher or lower note. The first and most natural mode occurs when the string in its oscillations produces a single arch; then it makes the slowest oscillations and gives out the deepest of all its possible tones, fundamental to all the rest. The next mode demands that the string produce two arches on the opposite sides [of the string's rest position] and then the oscillations are twice as fast, and now it gives out the octave of the fundamental sound.

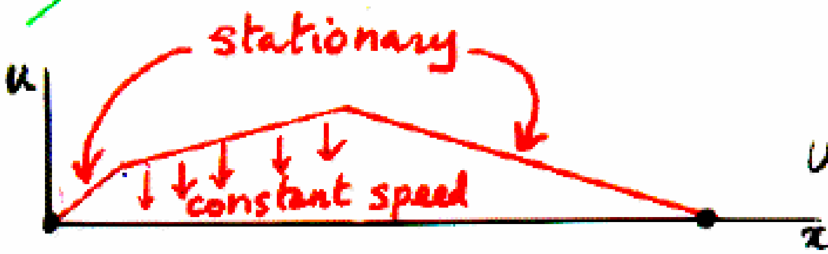
D'ALEMBERT $\quad u_{tt} = c^2 u_{xx}$, $u(t,x)$ analytic

EULER

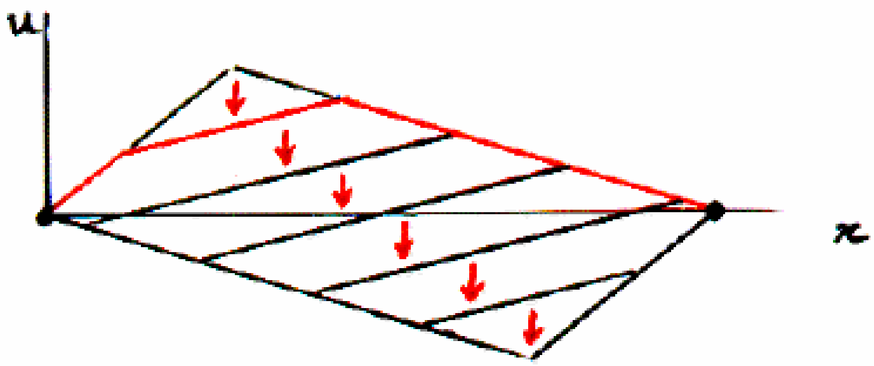
the plucked string of a harpsichord



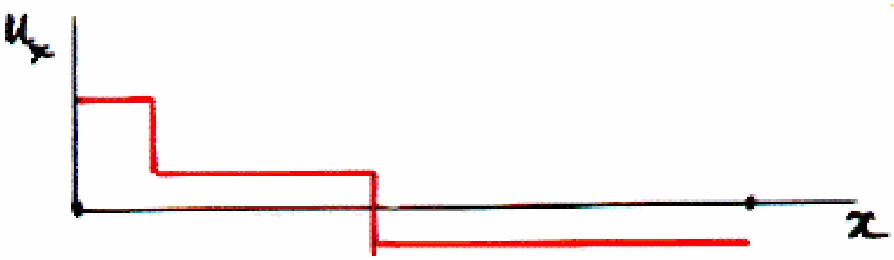
$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$



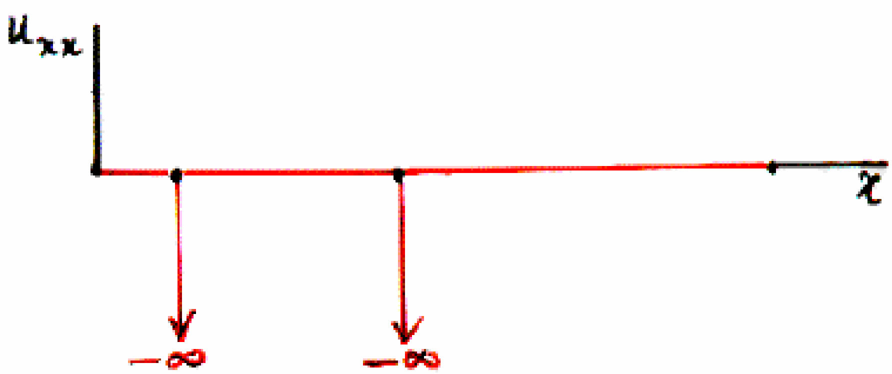
$u(t,x)$ at time t



one cycle



discontinuous first derivative



Dirac function second derivative.

1759: Euler

On October 2, 1759, Euler had written Lagrange: "I am delighted to learn that you approve my solution ... which d'Alembert has tried to undermine by various cavils, and that for the sole reason that he did not get it himself. He has threatened to publish a weighty refutation; whether he really will I do not know. He thinks he can deceive the semi-learned by his eloquence. I doubt whether he is serious, unless perhaps he is thoroughly blinded by self-love."

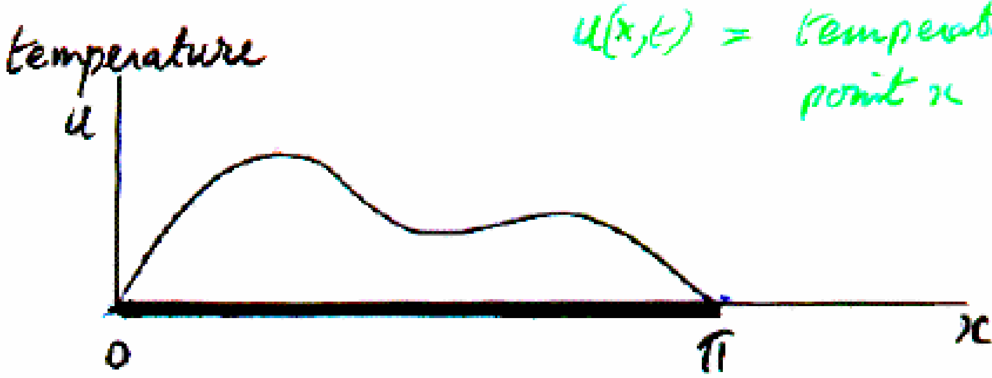
1753 Daniel BERNOULLI

We remark therefore that the string can make not only [Taylor] vibrations of the first, second and third kinds, and so on to infinity, but also any sum of these vibrations in all possible combinations. Meanwhile all these new curves and new types of vibration given by Messrs. d'Alembert & Euler are absolutely nothing more than sums of several kinds of of Taylor vibrations.

FOURIER (1807)

Solved the problem of how heat diffuses away from a bar, whose ends are kept at zero temperature.

$u(x,t)$ = temperature at point x at time t.



Heat equation: $u_t = \delta u_{xx}$, $\delta > 0$.

Boundary condition $u(0,t) = u(\pi,t) = 0$, $\forall t$.

Initial condition $u(x,0) = f(x)$.

Solution $$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\delta n^2 t} \sin nx,$$

where the Fourier coefficients a_n are determined by initial condition at $t=0$,

$$f(x) = \sum a_n \sin nx$$

and hence given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

THE INTUITION OF D'ALEMBERT & EULER

1. Not every continuous function in $[0,\pi]$ can be expressed as a sum of harmonics — FALSE

$$f(x) = \sum_{1}^{\infty} a_n \sin nx$$

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2. Anyway, given an arbitrary function f there is no way you could possibly calculate the coefficients a_n — FALSE (Fourier)

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3. Furthermore, if f is not periodic you could not express it as a sum of periodic functions in $[0,\pi]$. — FALSE

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4. The limit of a convergent series of analytic functions must be analytic. — FALSE

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5. Anyway, by the uniqueness of analytic continuation, if $\sum f_n$ converges to f in an interval, and if both f and all the f_n are analytic in a larger interval, then $\sum f_n$ must converge to f in the larger interval. — FALSE

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6. Quite apart from analyticity, the limit of a convergent series of continuous functions must be continuous. — FALSE requires uniform convergence

("Theorem" of Cauchy, 1823)

DANIEL BERNOULLI WAS RIGHT

CONCLUSIONS ABOUT THE CONTROVERSY

1. Daniel Bernoulli, D'Alembert & Euler each made major positive contributions.

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2. Bernoulli's was the deepest because it gave insight into music & harmony.

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3. Most of their intuitive criticisms of each other's work turned out to be wrong in retrospect.

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4. The controversy lasted 80 years until finally Dirichlet found the mathematical proof.

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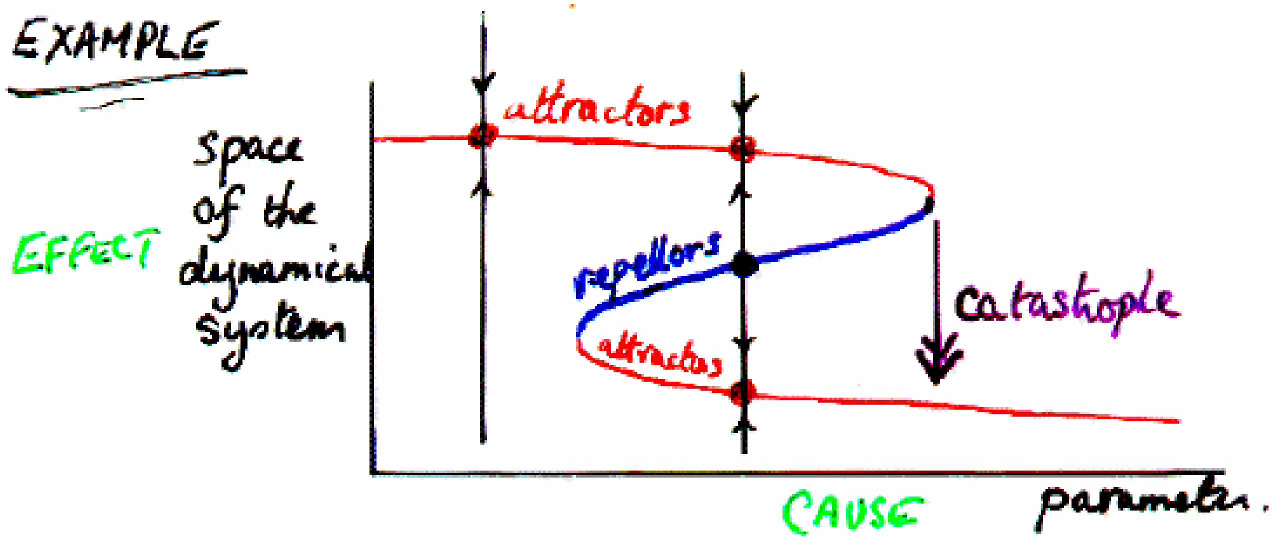
5. The controversy was mathematically very creative, because it led to the understanding of functions, differential equations, Fourier series, uniform convergence, & analysis.

CATASTROPHE THEORY

is concerned with multidimensional parametrised dynamical systems.

EXAMPLE



Continuous change of parameter ⇒ discontinuous change of attractor.

APPLICATIONS

Continuous cause ⇒ discontinuous effect

Elementary theory:

point attractors

∴ can use Lyapunov functions.

Thom classified the singularities of generic parametrised functions, which he called the elementary catastrophes.

Non-elementary theory:

cyclic & chaotic attractors

PRECURSORS

1945 Whitney's Theorem classifying stable singularities of maps $\mathbb{R}^2 \to \mathbb{R}^2$ (fold & cusp).

1956 Thom's extension to higher dimensions

1958 Thom awarded a Field's Medal

1966 Smale awarded a Field's Medal

CATASTROPHE THEORY CONTROVERSY

1968 Thom's Theorem classifying elementary catastrophes (with help from Malgrange & Mather)

1972 Thom's book: Structural stability & morphogensis

CONTROVERSY

1974 Zeeman's address to Vancouver Int. Congress of Maths.

1976 Zeeman's article in Scientific American

1977 Zahler & Sussman's article in Nature, + replies.

1977 Zeeman's book: Catastrophe theory

1978 Smale's review in Bulletin of Am. Math. Society

1980 Smale reprints the review in his book: The mathematics of time.

THE APPEAL OF CATASTROPHE THEORY

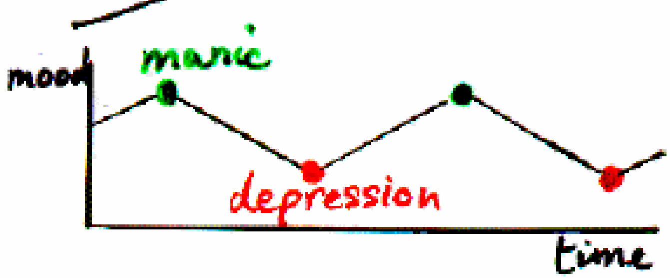
1. The elegance & depth of the pure mathematics.
2. The insight & overall grasp that the qualitative geometric approach gives to applications.
3. The simplicity of models: this is due to the theorems that guarantee smooth changes of coordinates into coordinates that are intrinsic to the application, and with respect to which the models have canonical polynomial equations.

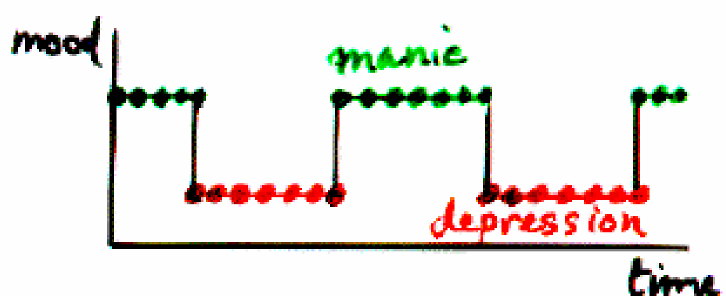
THE USEFULNESS OF THE WORD "CATASTROPHE"

It focuses attention upon discontinuities, and sudden jumps caused by smoothly changing parameters, thus enabling researchers to

a) recognise them
b) look for them
c) discuss them
d) design experiments to test them, and
e) process the data so as to reveal them.

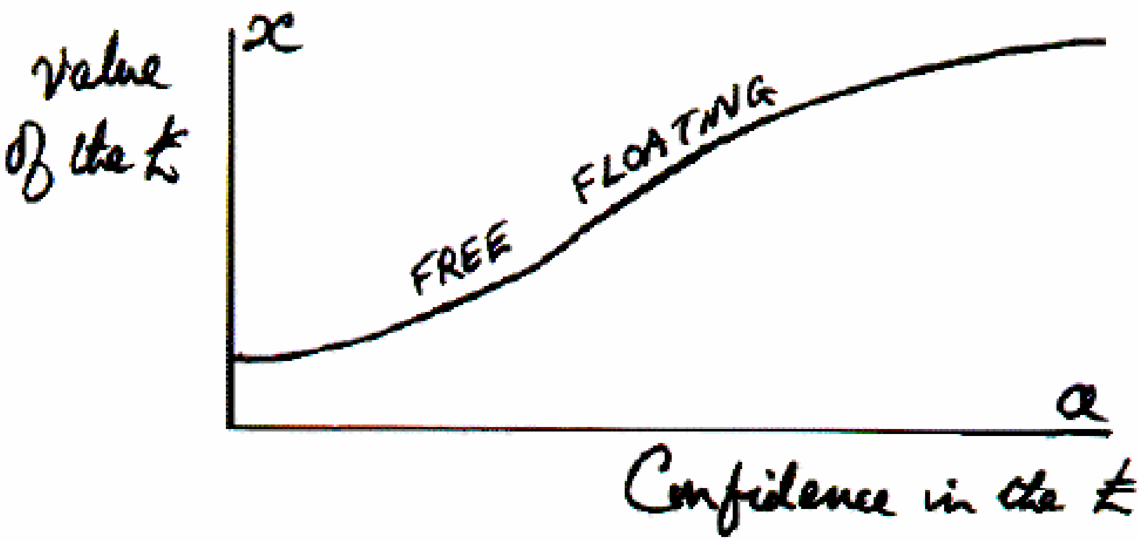
Example





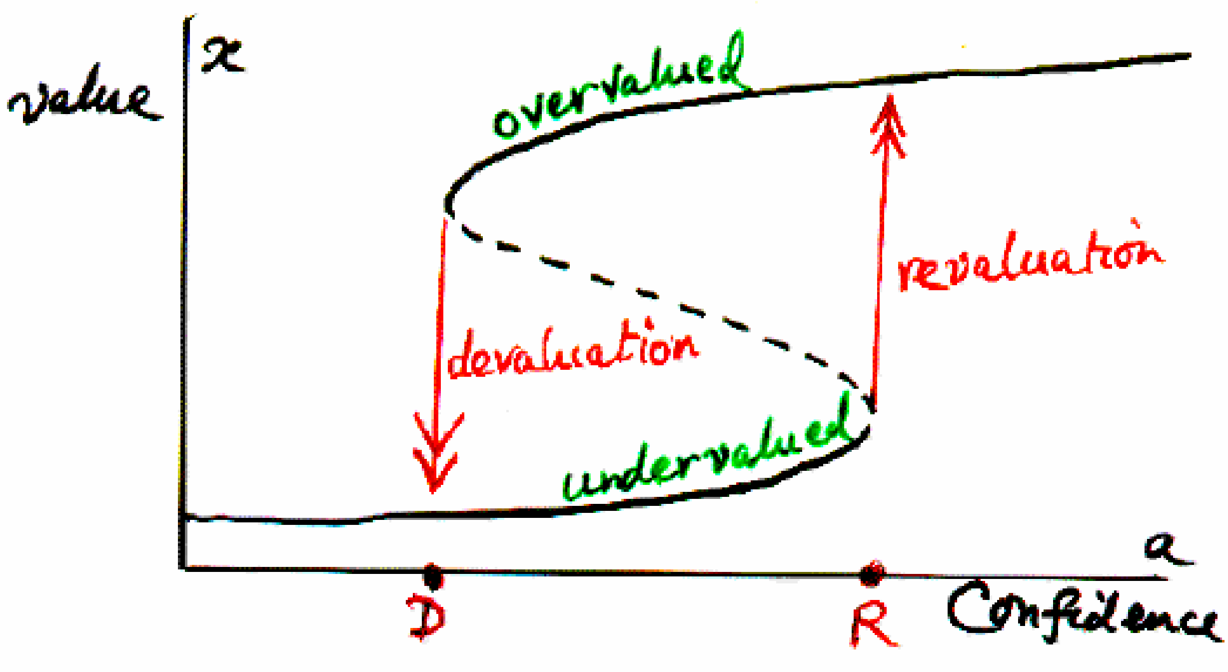
HOW THE VALUE OF THE £ DEPENDS UPON CONFIDENCE IN THE £.

① WITH A FLEXIBLE EXCHANGE RATE



② WITH A CONSTRAINED EXCHANGE RATE

(LIKE THE ERM, European Exchange Rate Mechanism).



③ A CUSP CATASTROPHE

THE GRAPH OF VALUE, x, OVER THE PARAMETERS CONFIDENCE, a, & FLEXIBILITY, b.

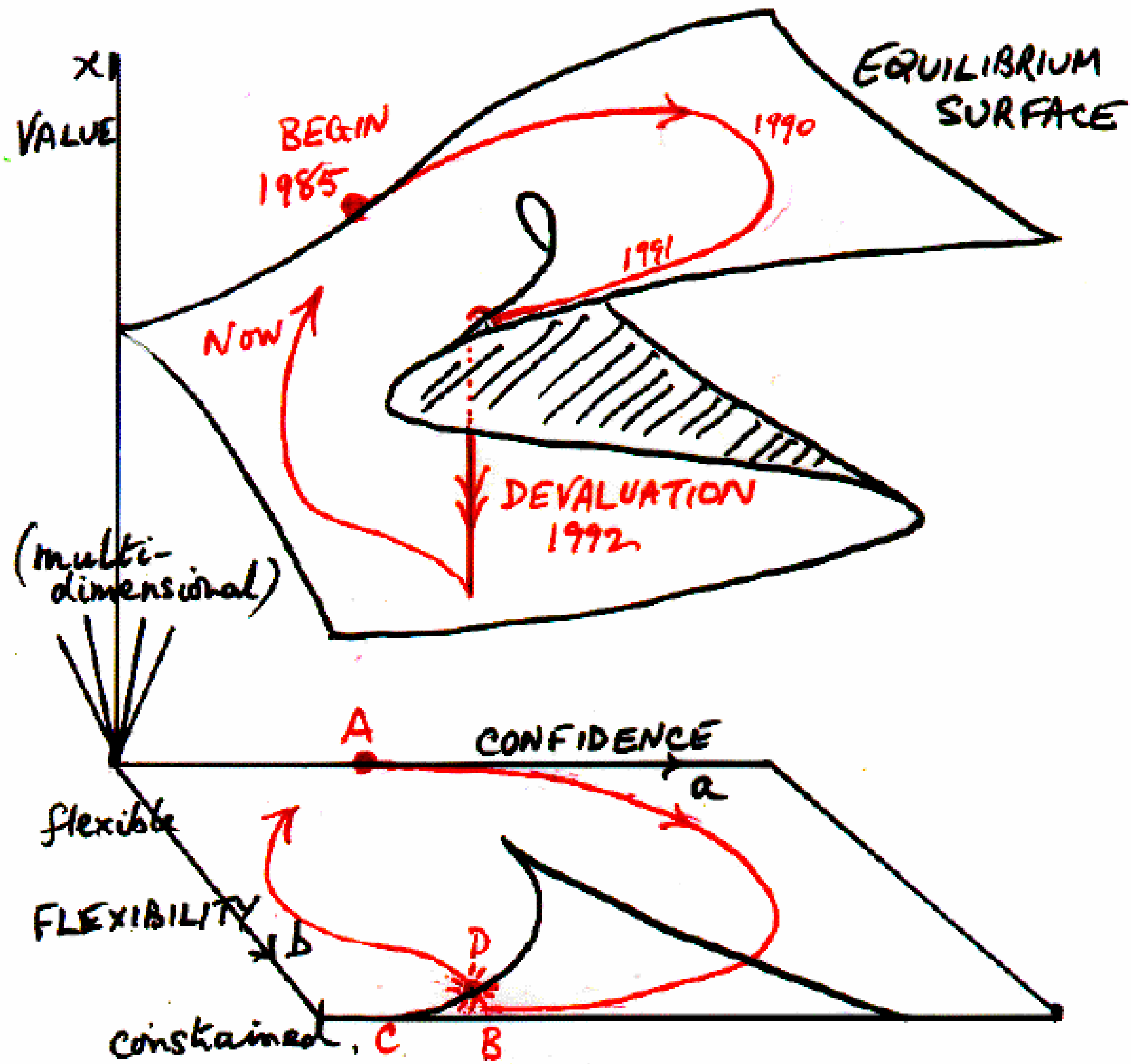


Figure 1

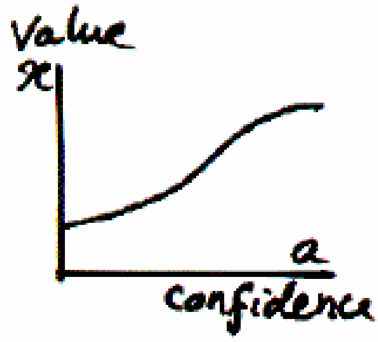


Figure 2

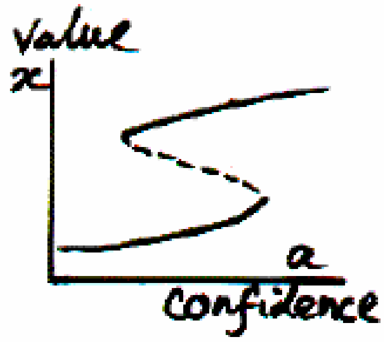
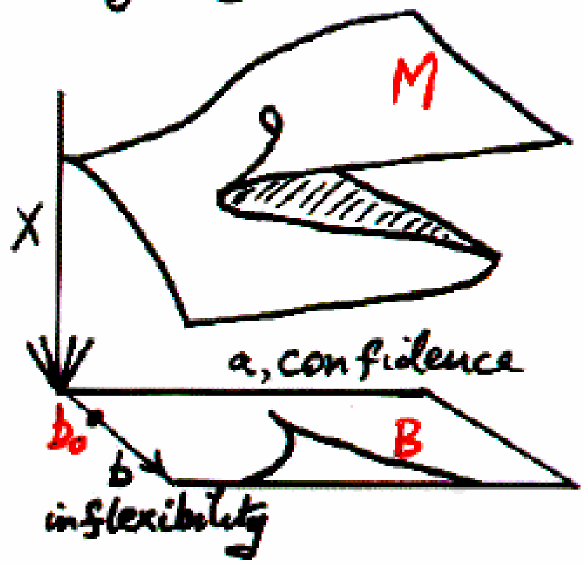


Figure 3



HYPOTHESES ABOUT M

HYPOTHESIS 1

The sections of M are equivalent to $\begin{cases} \text{Figure 1 if } b < b_0 \\ \text{Figure 2 if } b > b_0 \end{cases}$

HYPOTHESIS 2

M is the equilibrium set of a parametised dynamical system on X modelling the market, with parameters a, b, which has a generic parametrised Lyapunov function f.
The unshaded part of M consists of stable equilibria (minima of f), and the shaded part of unstable equilibria (maxima or saddles of f).

DEDUCTION By using Thom's deep classification theorem we can deduce that Figure 3 is a cusp catastrophe.

CRITICISMS A LA SMALE

CRITICISM ①

No justification is given for the model in terms of existing data or economic theory. It fits the caricature of a mathematician throwing a model to the economists to pick up & develop.

ANSWER ①

True, but intentional.

CRITICISM ②

Hypothesis 1 alone gives the structure of the surface in 3-dimensions without using any mathematics at all. There is no need to invoke "Thom's deep classification theorem", and to do so is not only misleading but also mystifying and intimidating to non-mathematicians

ANSWER ②

Criticism ② is an elementary mathematical mistake, because there is a simple counterexample showing Hypothesis 1 $\not\Rightarrow$ a cusp catastrophe.

CRITICISM ③

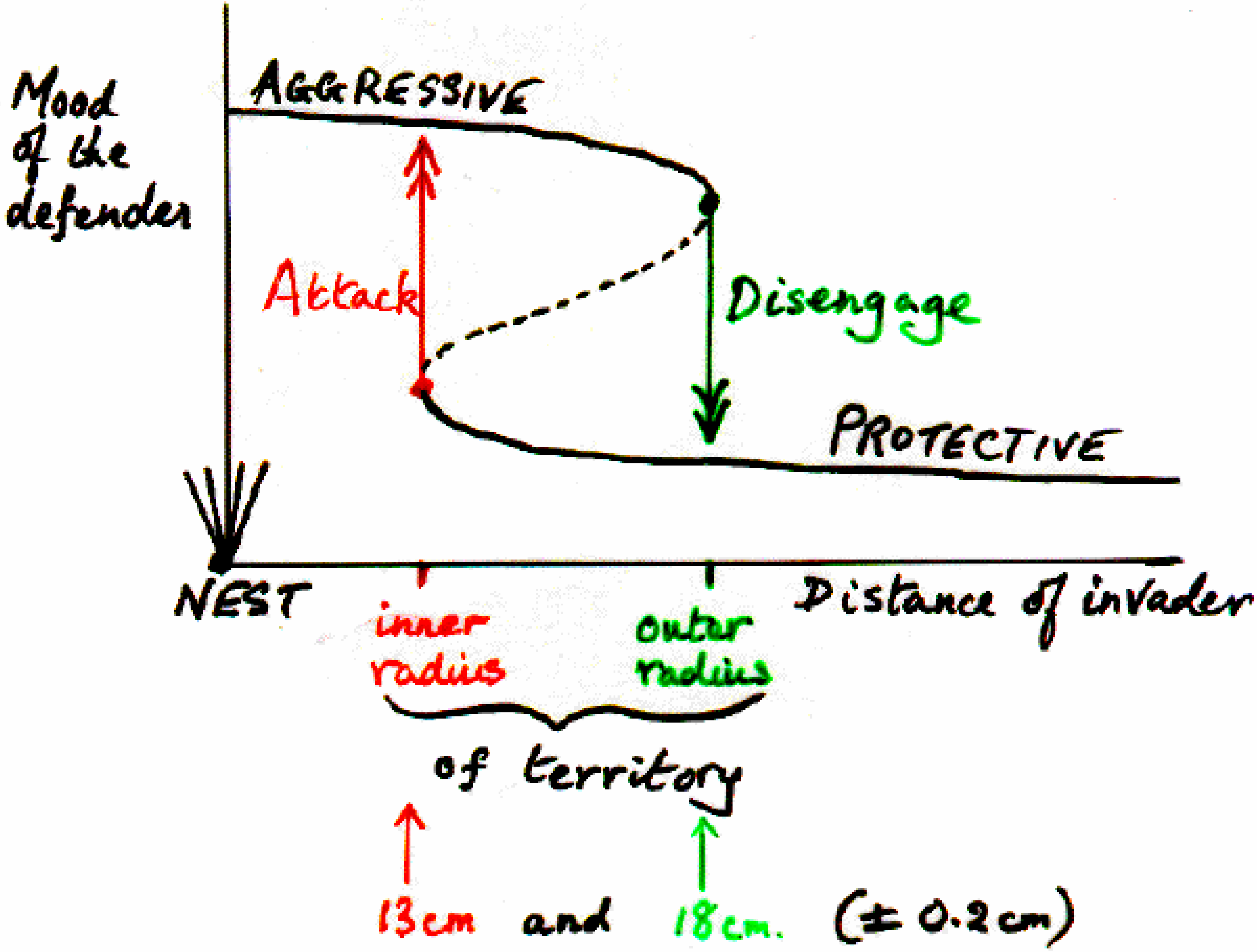
Anyway the cusp is not Thom's theorem but Whitney's theorem.

ANSWER ③

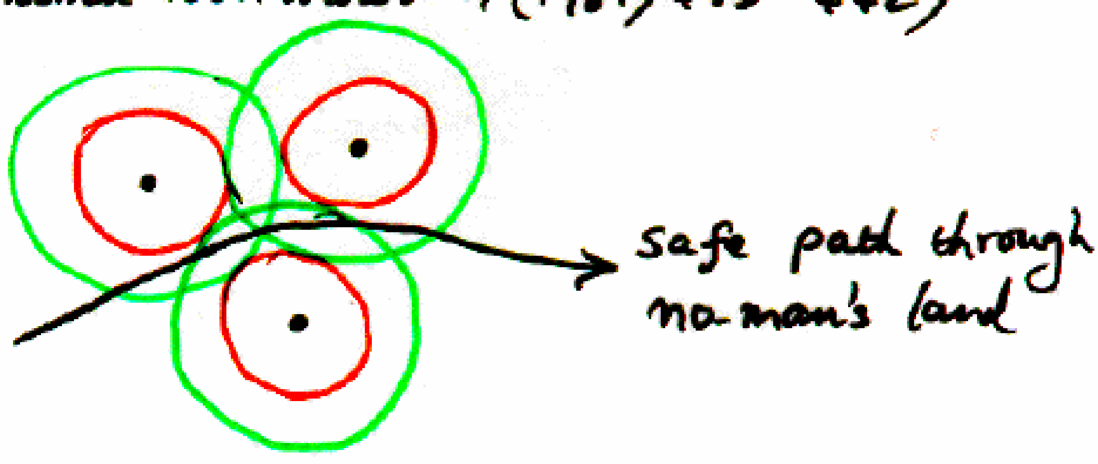
Criticism ③ is a serious mathematical mistake, revealing a misunderstanding of the difference between the two theories.

TERRITORIAL FISH

Zeeman: Scientific American (April 1976)



1980 P.W. COLGAN measured the territories of pumpkinseed sunfish in Lake Opinicon, Canada (Animal Behaviour 29 (1981) 433–442)



DEFINITION OF THE CANONICAL CUSP CATASTROPHE.

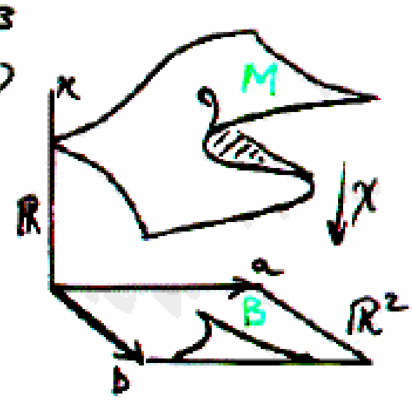
The parametrised Lyapunov function $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(a,b,x) = \tfrac{1}{4}x^4 - ax - \tfrac{1}{2}bx^2.$$

The parametrised dynamic on $\mathbb{R}$ is given by

$$\dot{x} = -f_x$$

The equilibrium surface M, $\subset \mathbb{R}^3$, is given by $f_x = x^3 - a - bx = 0$.



The map $\chi : M \to \mathbb{R}^2$ is induced by projection

The bifurcation set B, $\subset \mathbb{R}^2$, is obtained by eliminating x from $f_x = f_{xx} = 0$, giving the cusp $\quad 27a^2 = 4b^3.$

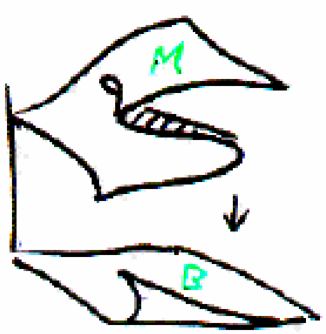
DEFINITION OF A CUSP CATASTROPHE.

Let $f' : \mathbb{R}^3 \to \mathbb{R}$ be another parametised Lyapunov function for some dynamic (not necessarily the gradient of f'). Suppose f' determines M', χ', B'.

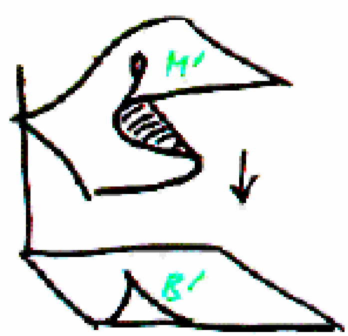
We call f' a cusp catastophe if it is parametised-function-equivalent to f, in other words $\exists$ diffeomorphisms α, β, γ such that the diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\alpha", "\text{diffeo}"'] \arrow[d, "\pi \times f"] & \mathbb{R}^3 \arrow[d, "\pi \times f'"] \\
\mathbb{R}^2 \times \mathbb{R} \arrow[r, "\beta", "\text{diffeo}"'] \arrow[d, "\pi"] & \mathbb{R}^2 \times \mathbb{R} \arrow[d, "\pi"] \\
\mathbb{R}^2 \arrow[r, "\gamma", "\text{diffeo}"'] & \mathbb{R}^2
\end{tikzcd}$$

It follows that χ, χ' are map-equivalent, in other words the diagram commutes:



$$\begin{tikzcd}
M \arrow[r, "\alpha|M", "\text{diffeo}"'] \arrow[d, "\chi"] & M' \arrow[d, "\chi'"] \\
\mathbb{R}^2 \arrow[r, "\gamma", "\text{diffeo}"'] & \mathbb{R}^2
\end{tikzcd}$$



$\therefore$ the cusp point on B maps to a cusp point on B'.

$\therefore$ B' contains a cusp.

COUNTEREXAMPLE showing HYP ① $\not\Rightarrow$ A CUSP CATASTROPHE.

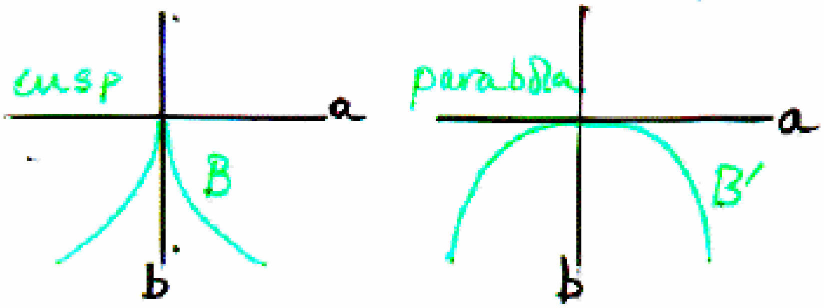
It suffices to find f' such that M' satisfies Hypothesis 1 but B' does not contain a cusp.

The trick is to replace a by a^3 in the formula for f.

Let $$f' = \frac{1}{4}x^4 - a^3x - \frac{1}{2}bx^2.$$

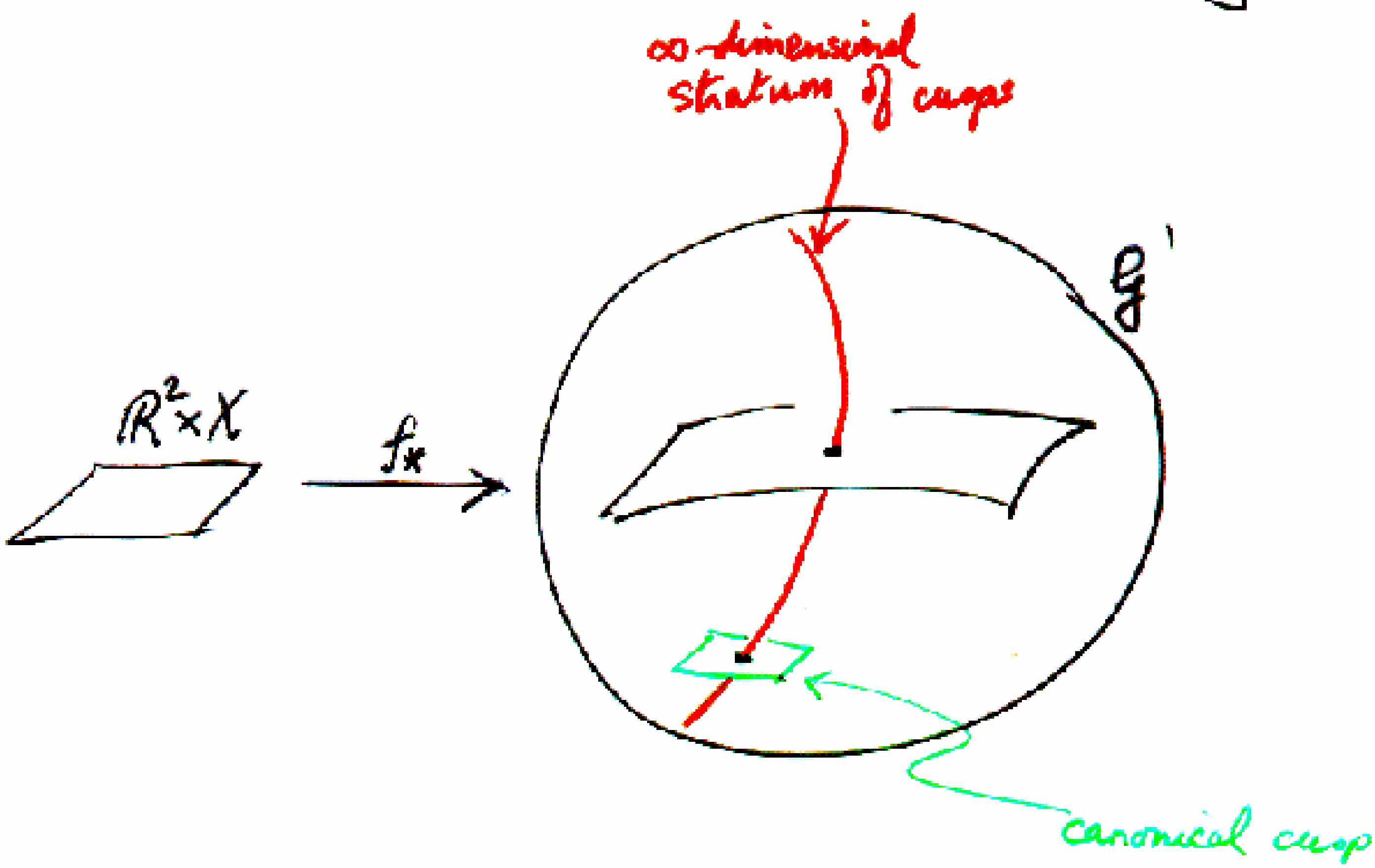
Then B' is given by $$27a^6 = 4b^3$$

$$\therefore\ 3a^2 = 4^{1/3}b\,,$$ parabola



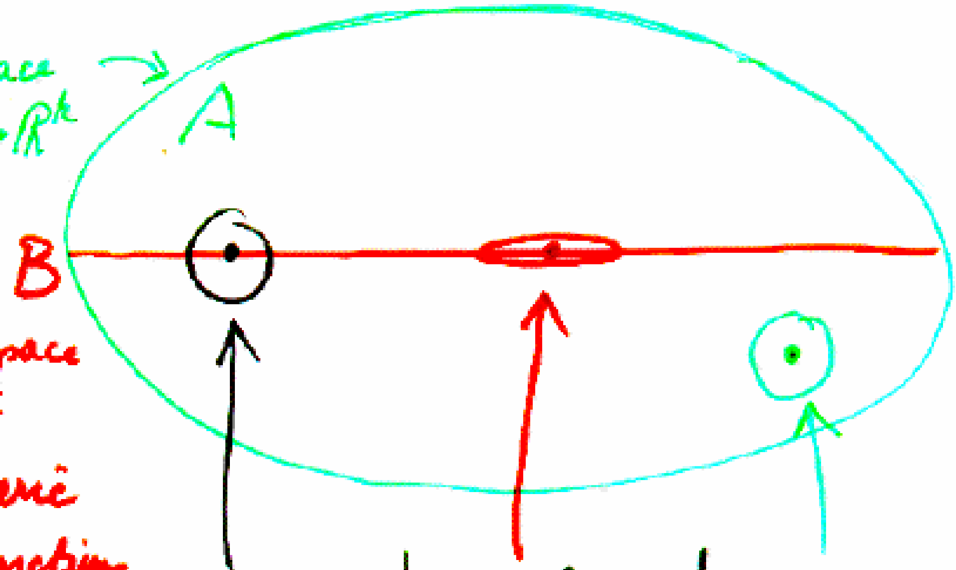
DEFINITION OF GENERIC.

A parametrised function f on X with parameters $\mathbb{R}^2$ is called <u>generic</u> if the induced map f_* into the space $\mathcal{G}$ of germs of functions on X is transversal to the natural stratification of $\mathcal{G}$.



DIFFERENCE BETWEEN WHITNEY THEORY & THOM THEORY.

∞-dimensional space of all maps $g : \mathbb{R}^k \to \mathbb{R}^k$



∞-dimensional subspace of maps $M \to \mathbb{R}^k$ arising from generic parametrised functions.

		$g \in B$ stable in A	$g \in B$ stable in B unstable in A	$g \notin B$ stable in A
WHITNEY	Stable singularities	✓	✗	✓
THOM	Elementary catastrophes	✓	✓	✗
k = number of parameters	1	fold	—	—
	2	cusp	—	—
	3	swallowtail	elliptic umbilic, hyperbolic umbilic	—
	4	butterfly	parabolic umbilic	$\Sigma_{\pm}$
	⋮	⋮	⋮	⋮

CONCLUSIONS ABOUT THE CATASTROPHE THEORY CONTROVERSY

1. The controversy was relatively short-lived because the underlying mathematics had already been established and was uncontroversial.

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2. The controversy was mainly about applications, and between mathematicians rather than between experts in the fields of those applications.

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3. The controversy was similar to that between d'Alembert, Euler & Bernoulli in that both Thom & Smale made major positive contributions to topology & dynamical systems, but Smale's negative contribution was not so great.

EPILOGUE

1990. EXTRACT FROM ZEEMAN'S BANQUET ADDRESS AT THE SMALEFEST CELEBRATING SMALE'S 60th BIRTHDAY

"Of course the real evidence for the excellence of Steve's mathematical judgement is the number of mathematicians world-wide who now follow his taste. In fact I have only known him to make one serious mathematical error of judgement, and that was his opinion of catastrophe theory."