Addendum to E.C. Zeeman : Introduction to Knot-Theory.

Letter from Zeeman to Professor D. Gokhman.

Dear Dmitry,

Many thanks for your letter. I agree that the proof of Theorem 8 is in a different category to those of Theorems 1-7, but in its context it is rigorous. Let me explain, & then ~~give~~ justify my approach.

Theorems 1-7 lie in combinatorial knot theory, whereas the proof of Theorem 8 lies in topological knot theory.

In combinatorial knot theory we define a tame knot to be an embedding $S^1 \subset R^3$ that has a projection consisting of a finite number of arcs & crossings; define two tame knots to be combinatorially equivalent if it is possible to go from a projection of one to a projection of the other by a finite sequence of elementary moves.

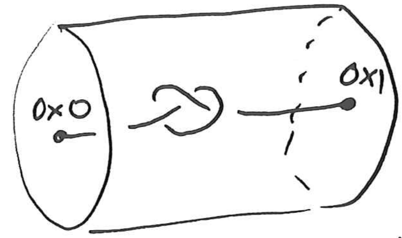
In topological knot theory we define a knot to be an embedding $S^1 \subset R^3$; define two knots to be topologically equivalent if $\exists$ an orientation-preserving homeomorphism of R^3 onto itself throwing one onto the other.

Note: (i) Both theories agree on tame knots, because one can show that two tame knots are combinatorially equivalent if & only if they are topologically equivalent.

(ii) The topological theory is larger because it also contains wild knots, any of whose projections must have an infinite number of crossings, and which are not equivalent to any tame knot.

To construct the semi-group of knots we introduce the analogous topological theory of knotted arcs as follows. Let I denote the unit interval, D the unit disk (in the complex plane), & E the solid cylinder $E = D \times I$. Define a knotted arc A to be an embedding $I \subset E$ that maps the interior to the interior, and the boundary $\{0, 1\}$ to $\{0\times 0, 0 \times 1\}$.

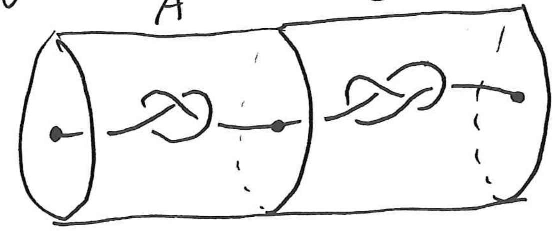
E



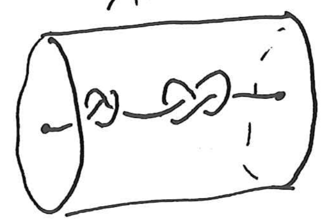
Define two to be topologically equivalent if $\exists$ a homeomorphism of E onto itself, keeping the boundary pointwise fixed, and throwing one onto the other.

Define the product $A \times B$ by juxtaposition & contraction:

A B



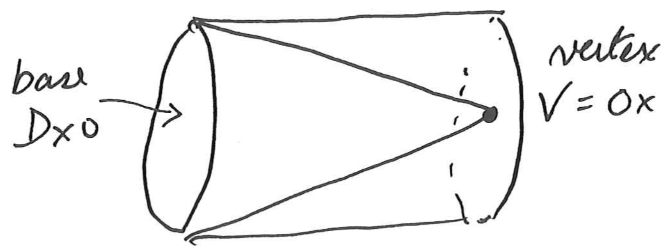
A×B



Theorem 8 stated that $A \times B = 1 \Rightarrow A = 1$ & $B = 1$.

For the proof we constructed an arc X, as follows.

Let $C \subset E$ be the solid cone base $D \times 0$ and vertex $V = 0 \times 1$.

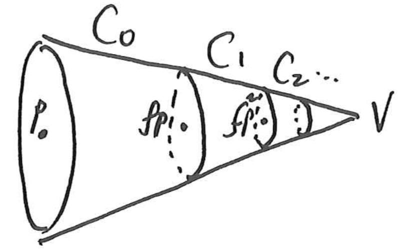


Let $f : C \to C$ be linear contraction by $\frac{1}{2}$ towards V.

Let $C_0 = \text{closure}(C - fC)$

$C_n = f^n C_0, \quad n > 0.$

Then $C = \bigcup_0^\infty C_n \cup V.$



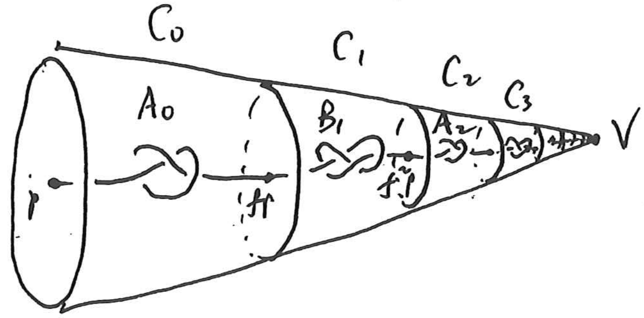
Let $p = 0 \times 0$.

Choose a copy A_0 of A in C_0 running from p to fp.

& a copy B_1 of B in C_1 running from fp to $f^2 p$.

$$\text{Let } A_{2n} = f^{2n} A_0, \subset C_{2n}, \quad n > 0$$

$$B_{2n+1} = f^{2n} B_1, \subset C_{2n+1}, \quad n > 0.$$



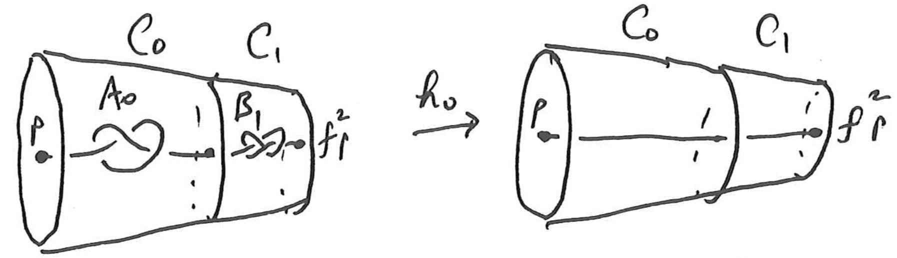
$$\text{Let } X = \bigcup_{n=0}^{\infty} (A_{2n} \cup B_{2n+1}) \cup V$$

$$= A_0 \times B_1 \times A_2 \times B_3 \times \cdots$$

Then X is an arc because $C_n \to V$ as $n \to \infty$, & $X \subset C \subset E$.

Note X is not tame, because any projection of X contains an infinite number of crossings. Nevertheless we shall show that it is unknotted.

Since $A \times B = 1$, by hypothesis, $\exists$ a homeomorphism h_0 of $C_0 \cup C_1$ onto itself keeping the boundary pointwise fixed & throwing $A_0 \times B_1$ onto the interval $p(f^2 p)$



Define $h: E \to E$ by $\begin{cases} h | C_{2n} \cup C_{2n+1} = f^{2n} h_0 f^{-2n}, & n \geq 0 \\ h | V \cup (E - C) = 1 \end{cases}$

Then h is continuous at V because $C_n \to V$ as $n \to \infty$.

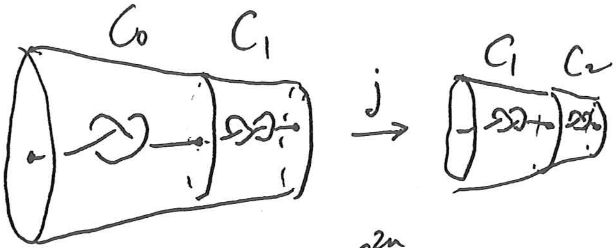
Therefore h is a homeomorphism keeping the boundary pointwise fixed & throwing X onto the interval pV.

$\therefore X = 1$

By commutativity $\exists$ a homeomorphism $j: C_0 \cup C_1 \to C_1 \cup C_2$ such that $j |$ boundary $= f$, & throwing $A_0 \times B_1$ onto $B_1 \times A_2$.

Define the homeomorphism $k: E \to E$ by

$$\begin{cases} k | C_0 = 1 \\ k | C_{2n+1} \cup C_{2n+2} = f^{2n+1} h_0 j^{-1} f^{-2n}, & n \geq 0 \\ k | V \cup (E - C) = 1 \end{cases}$$



$$\begin{tikzcd}
C_0 \cup C_1 \arrow[r, "j"] \arrow[d, "h_0"] & C_1 \cup C_2 \arrow[r, "f^{2n}"] \arrow[d] & C_{2n+1} \cup C_{2n+2} \arrow[d, "k"] \\
C_0 \cup C_1 \arrow[r, "f"] & C_1 \cup C_2 \arrow[r, "f^{2n}"] & C_{2n+1} \cup C_{2n+2}
\end{tikzcd}$$

Then k keeps the boundary pointwise fixed & throws X onto $A_0 \times I$.

$\therefore X = A.$

$\therefore A = 1.$

Similarly $B = 1$.

Remark 1. In the above proof the definition of the arc X & the homeomorphisms h, k are continuous at V because $C_n \to V$ as $n \to \infty$, & hence all three are well defined.

Remark 2 In the proof

$$\begin{aligned}
A &= A \times I && (1) \\
&= A \times I \times I \times I \times \cdots && (2) \\
&= A \times (B \times A) \times (B \times A) \times (B \times A) \times \cdots && (3) \\
&= A \times B \times A \times B \times A \times B \times A \times \cdots && (4) \\
&= (A \times B) \times (A \times B) \times (A \times B) \times \cdots && (5) \\
&= I \times I \times I \times \cdots && (6) \\
&= I && (7)
\end{aligned}$$

each of the steps is rigorous because each line is well-defined, and

step (1) is geometrical trivial.

steps (2) & (7) depends on the fact that an interval is homeomorphic to the union of a countable number of juxtaposed intervals, each half the length of its predecessor, together with the end point V.

steps (3) & (6) depend on ~~the~~ homeomorphism h, k

steps (4) & (5) depend on the fact that the geometrical representation of a countable product is independent of associativity, and so the brackets are notational, & can be added or removed without changing the geometrical meaning.