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Non-Embeddable Functions with a Fixpoint of Multiplier 1

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1. Introduction

An analytic function $f(z)$ is said to have a fixpoint $\xi \neq \infty$ of multiplier 1 if $f(\xi)=\xi, f'(\xi)=1$. Without loss of generality we may put $\xi=0$, so that the function has an expansion

(1) $$f(z)=z+\sum_{k=2}^{\infty} a_k z^k$$

convergent in some neighbourhood of 0. We shall assume $a_2 \neq 0$. Now it has been shown (e.g. in [*1*]) that there is for every complex s a unique formal iterate

(2) $$f_s(z)=z+\sum_{2}^{\infty} a_k(s)\, z^k, \qquad a_2(s)=s\, a_2,$$

where the $a_k(s)$ are well-defined polynomials in s determined by comparing coefficients in the formal identity

(3) $$f \circ f_s(z)=f_s \circ f(z).$$

For positive integral values of $s=n$ (say), the series (2) is the same as that of the n-th iterate of $f(z) \equiv f_1(z)$. By analogy the $f_s(z)$ are in general called fractional iterates.

The series (2) does not necessarily have a positive radius of convergence for each s; in fact, as was shown in [*1*] the values s corresponding to a positive radius of convergence either fill out the whole complex plane or form a discrete one or two dimensional lattice. In the former case one may call $f(z)$ embeddable (in a continuous group of analytic iterates (2)). SZEKERES [*6*] and BAKER [*2*] showed that if (1) is the expansion of a function entire or even meromorphic in the plane, then $f(z)$ is not embeddable in this sense except in the single case

$$f=\frac{z}{1+a z},$$

a constant.

Recently RAN [*5*], using a method based on work of LEWIN ([*3*, *4*]) has shown that

$$f(z)=\frac{z}{\sqrt{1+z}}=z-\frac{1}{2} z^2+\frac{3}{8} z^3+\cdots$$

is not embeddable. This is the first example of a non-embeddable algebraic function other than those obtained as the inverses of polynomials and rational functions.

In this note we use a simpler version of the method of [2] to prove that a large class of functions is non-embeddable:

Theorem. *If the expansion*

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \tag{1}$$

is convergent in some neighbourhood of the origin, if

(a) *the analytic continuation of $f(z)$ is possible without restriction in the Riemann sphere punctured in a countable set of points and gives rise to a finitely many valued function there, and if*

(b) $a_2 \neq 0$, $a_2^2-a_3 \neq 0$,

then the series (1) *is not embeddable in the sense defined above.*

In particular, if (1) is the expansion of an algebraic function and (b) is satisfied, then it is not embeddable. This case includes RAN's example, for which $a_2=-\frac{1}{2}$, $a_3=\frac{3}{8}$.

The theorem also includes most of the meromorphic functions proved non-embeddable in [2], and achieves this result in a simpler way at the expense of omitting those non-bilinear meromorphic functions which fail to satisfy (b).

The theorem also treats many functions (e.g. meromorphic functions of algebraic functions) not previously considered in connexion with this problem.

In [1] it was pointed out that one can construct embeddable functions and the corresponding one parameter groups with expansion (2) corresponding to an arbitrary infinitesimal transformation. Using this we show in section 3 that one can find embeddable series (1) which satisfy (b) of our theorem and thus cannot have the property (a). We also show that if (b) is dropped one can find embeddable functions having the property (a); indeed one can find embeddable algebraic functions.

2. Proof of the Theorem

It is convenient to transfer the fixpoint to ∞. If we change variables in the transformation $z_1=f(z)$ by putting $z=k/t$, $z_1=k/t_1$ and choose k so that $-k a_2=1$ we obtain instead of (1) the transformation

$$t_1=t+1+\sum_{1}^{\infty} b_k t^{-k}=g(t) \tag{4}$$

which has a fixpoint at ∞. We note that

$$b_1=(a_2^2-a_3)/a_2^2. \tag{5}$$

The same change of variables applied to (2) turns $f_s(z)$ into

$$t_s=t+s+\sum_{1}^{\infty} b_k(s)\, t^{-k}=g_s(t), \tag{6}$$

where $g_s(t)$ form the unique family of formal series commuting with (4). The series (1) is embeddable precisely when the series (6) converges for some $t \neq \infty$ for every s.

From now on $g_s(t)$ will denote the series (6) and $g = g_1$ will be assumed convergent for $|t| > R$.

We quote the following results from [1, 6]:

Lemma 1 [1, p. 272]. *If the region*

$$\mathfrak{D}(K) = \bigcup_{-(\pi/4) \leqq \alpha \leqq (\pi/4)} \mathfrak{C}(\alpha, K),$$

where $\mathfrak{C}(\alpha, K)$ *is the half-plane* $\{z \mid \mathrm{Re}(z e^{-i\alpha}) > K\}$. *then for all sufficiently large* $K(>R)$, $g_n(z)$ *is regular,*

$$g_n(z) \in \mathfrak{D}(K), \qquad n = 1, 2, \ldots \tag{7}$$

and

$$\mathrm{Re}\, g_n(z) \to \infty \qquad as\ n \to \infty \tag{8}$$

for all z *in the closure* $\overline{\mathfrak{D}}(K)$ *of* $\mathfrak{D}(K)$.

By [1, 273 (21)] (8) holds uniformly on any compact subset of $\mathfrak{D}(K)$.

Lemma 2 [1, p. 273]. *For all sufficiently large* K *the domain* $\mathfrak{D}(K)$ *of Lemma 1 has the properties:*

$$A(t) = \lim_{n \to \infty} \{g_n(t) - n - b_1 \log n\}, \tag{9}$$

(*where* b_1 *is as in* (4)) *exists uniformly for* $t \in \mathfrak{D}(K)$; *moreover* $A(t)$ *is regular and schlicht in* $\mathfrak{D}(K)$ *and* $A'(t) \to 1$ *uniformly as* $t \to \infty$ *in* $\mathfrak{D}(K)$. *One has*

$$A(g_n(t)) = A(t) + n \qquad for\ t \in \mathfrak{D}(K). \tag{10}$$

Lemma 3 [6, § 2]. *If the series* (6) *have a positive radius of convergence for every* s, *then* $b(t) = A'(t)$ *is regular in a full neighbourhood of* $t = \infty$ *and has an expansion*

$$b(t) = 1 - b_1 t^{-1} + \sum_2^{\infty} \beta_k t^{-k} \tag{11}$$

which may be calculated from

$$b \circ g(t) = b(t)/g'(t).$$

We next prove

Lemma 4. *If the series* (6) *have a positive radius of convergence for each* s, *and if there exist a fixed* $R_1 > R > 0$ *and arbitrarily large integers* n *such that the total number of branches obtained by analytic continuation of* $g_n(t) = t + n + \sum b_k(n)\, t^{-k}$ *within* $|t| > R_1$ *is finite, then the coefficient* b_1 *in* (4) *and* (5) *is zero.*

Proof. Choose $K > R_1$ and so large that Lemma 2 holds and that (Lemma 3) $A'(t)$ is regular in $|t| > K$. We may suppose $A'(t)$ as close to 1 as we wish in $\mathfrak{D}(K)$ by choosing K large enough. Now $t \to w = A(t)$ maps $\mathfrak{D}(K)$ univalently and conformally onto a region $\mathfrak{C}$ of the w-plane lying to the right of a curve (of the same general appearance as the boundary of $\mathfrak{D}(K)$) which approaches ∞ in the directions $\arg w = \pm 3\pi/4$. $\mathfrak{C}$ contains a half-plane H: $\operatorname{Re} w > K'$.

We now assume $b_1 \neq 0$. Choose a fixed $R' > K$. Let γ be the segment $[R', \infty)$ of the real axis, described from ∞ to R', and let β be the circle $|t| = R'$ described in the positive sense, starting and finishing at R'. Now $A'(t)$ is regular on β and γ and so $A(t)$ may be continued analytically from ∞ along $\gamma\,\beta$ or indeed along $\gamma\,\beta^k$ for any positive or negative integer k. If $b_1 \neq 0$ the function $A(t)$, although continuable without restriction on $\gamma\,\beta^k$, will have an infinity of values and its value will increase by $-2\pi i b_1$ for each circuit of β.

Now for any integer n we have $g_n(t) = A_{-1}(A(t)+n)$ on γ near ∞, by Lemma 2. If we choose n sufficiently large, then as t describes $\gamma\,\beta$ the values of $w = A(t)+n$ form a set S which lies in the right half-plane H defined above.

Consider the sets $S + 2\pi i b_1 k$ and $S - 2\pi i b_1 k$, where S is as above and k is a positive integer. Then since S lies in the half-plane H we must either have $S + 2\pi i b_1 k \subset H$ for all k or $S - 2\pi i b_1 k \subset H$. Suppose, say, that $S - 2\pi i b_1 k \subset H$ for all positive k. As t describes β yet again, starting from R' at the end of $\gamma\beta$ and proceeding in the positive direction, then the values of $A(t)+n$ differ by $-2\pi i b_1$ from the values taken during the previous circuit. But these values lie in $S - 2\pi i b_1$ and so in H. Similarly as we describe $\gamma\beta^k$, for any integer $k > 0$, the values of $A(t)+n$ lie in

$$\bigcup_{j=0}^{k} (S - 2\pi\, i\, b_1\, j)$$

and so in H.

We note that the values of $A(R')+n$ corresponding to the different continuations along $\gamma\beta^k$, $k = 0, 1, 2, \ldots$ form an infinite set $w_0 - 2\pi i b_1 k$, $k = 0, 1, 2, \ldots$ in H. We now consider the continuation of $g_n = A_{-1}(A(t)+n)$ along $\gamma\beta^k$, $k > 0$, and note that during this continuation $A(t)+n$ remains in H, where $A_{-1}(w)$ is univalent and regular. Thus for each $k > 0$ we obtain a different continuation of $g_n(t)$ to R' along $\gamma\beta^k$ in $|t| \geqq R'$, with value $g_n(R') = A_{-1}(w_0 - 2\pi i b_1 k)$.

But we may assume n is one of the integers, whose existence is assumed in the statement of the Lemma, such that only a finite number of branches may be obtained by continuation within $|t| > R_1$ and a fortiori by continuation within a part of $|t| \geqq R' (> K > R_1)$. Thus we have a contradiction unless $b_1 = 0$.

We may remark that if in the above it had been the sets $S + 2\pi i b_1 k$, $k = 0, 1, 2, \ldots$ which had been contained in H, we should simply have considered continuations around the paths $\gamma\beta^{-k}$ and the proof would have been otherwise unaltered.

This concludes the proof of Lemma 4.

Lemma 5. *Let F be the class of analytic functions $f(z)$ with the property: if the Riemann sphere S is punctured in a suitably chosen and at most countably infinite set of points $E = E(f)$ then some branch of $f(z)$ meromorphic at a point* *of $S-E$ can be continued unrestrictedly in $S-E$ with meromorphic character and gives rise to a finitely many valued function $f_E(z)$.*

If $f(z)$ and $g(z)$ belong to F then so does $f(g(z))$, and hence so also does the iterate $f_n(z)$ for any integer $n \geqq 1$. This statement is true for any choice of the initial branches of f and g so long as $w=g(z)$ is analytic at the base point z_0 and $f(w)$ is analytic at $w=g(z_0)$.

Proof. We show first that $f_E(z)$ contains all elements of the analytic configuration A of $f(z)$ lying over points of $S-E$. Let $q \in S-E$ and $P(z, q)$ be an arbitrary element of A over q. Then if $P(z, p)$ was the initial branch of $f(z)$ used to generate $f_E(z)$ by continuation, we know, since $P(z, p) \in A$ that there is a curve C running from p to q on which $P(z, p)$ may be continued meromorphically to $P(z, q)$. Now the continuation of $P(z, p)$ is meromorphic on the compact set C and hence also on the set D of those points whose spherical distance from C is less than ε, for some $\varepsilon > 0$. We can find a curve C' running from p to q in $D-E$. It suffices to show this in the case when neither p nor q is ∞. We note that, since D is connected, p can be joined to q by a polygon P whose sides are parallel to the x or y axis. Now only a countable number of the lines $x=\text{const}$ or $y=\text{const}$ can meet E, so by small displacements of the sides of P we may assume that no sides of P (except perhaps the first and last) meet E. To deal with the terminal sides we may have to rotate them by an arbitrarily small amount to remove points of E. In this way C' may be constructed as the modification of P. The continuation of $P(z, p)$ along C' in $S-E$ to q results in the same element $P(z, q)$ as does continuation along C. This establishes the statement made at the beginning of the paragraph.

We may easily deduce that the number of branches of $f(z)$ (and hence $f_E(z)$) over any point $p \in S-E$ is the same number, m say, independent of the choice of p. Suppose the corresponding number of branches of $g(z)$ in $S-G$, where $G=E(g)$, is n.

We next observe that for any fixed p the set of z such that for some branch of $g(z)$ we have $g(z)=p$ contains only isolated points and is therefore countable. Thus the set H of z for which some branch of $g(z)$ is meromorphic and $g(z)$ lies in $E=E(f)$ is a union of countable sets and thus countable. Thus $K=H \cup G$ is countable.

Next take an arbitrary point p in $S-K$ and take any branch of $g(z)$ there, given say by the function element $w=g=\varphi(z)$ meromorphic at p. Then $\varphi(p)=q \in S-E(f)$. Thus we may take any branch of $f(z)$ given, say, by a function element $f=\psi(z)$ meromorphic at q. Thus one may substitute φ in ψ and obtain a branch of $f(g)$ given near $z=p$ by the meromorphic element $\psi(\varphi(z))$. We show that this element may be continued unrestrictedly in $S-K$ and gives rise to at most mn branches over any given point.

Let C be any path in $S-K$ leading from p to some point t. The element $w=g=\varphi(z)$ may be continued meromorphically from p to t along C and the values w describe a curve $\varphi(C)$ in $S-E$. The continuation of $\varphi(z)$ at t is one of the branches of $g_G(z)$ and the terminal point of $\varphi(C)$ is the corresponding value of $u=g_G(t)$. The function element $\psi(w)$ continues into a branch of $f_E(w)$ over u as w moves along $\varphi(C)$ from $\varphi(p)$ to u. Thus, as z moves along C from p to t, the function $\psi(\varphi(z))$ is meromorphically continued into one of the composites $f_E(g_G(z))$ where $g_G(z)$ is one of the n elements of the analytic configuration of g lying over t, and for a given $g_G(z)$ such that $g_G(t)=u$, f_E is one of the m elements of the analytic configuration of f lying over u. Thus one can obtain at most mn different branches of $f(g)$ in this way. This completes the proof of the lemma.

Proof of the Theorem. If we assume that in (1) $f(z)$ satisfies condition (a) of the theorem and is embeddable, so that in (4) $g(t)$ satisfies condition (a) and is embeddable, then by Lemma 5 we know that for any fixed integral $s=n$ the $g_n(t)$ of (6) has at most a finite number of branches. To be exact the lemma tells us that for some countable set E, $(g_n)_E$ is finitely many valued, but the opening part of the proof of Lemma 5 shows that $(g_n)_E$ contains all the elements of the analytic configuration of g_n over points of $S-E$ so we know that $g_n(t)$ has a finite number of branches obtainable by *any* analytic continuation. Thus Lemma 4 applies and we conclude that

$$b_1=(a_2^2-a_3)/a_2^2=0, \quad \text{i.e.} \quad a_2^2-a_3=0.$$

This concludes the proof of the theorem.

3. Counterexamples

a) Embeddable Functions Satisfying $a_2 \neq 0$, $a_2^2-a_3 \neq 0$

It was pointed out, for example in [*1*, p. 289], that if

$$I(z)=a_2 z^2+\sum_3^\infty b_n z^n$$

is regular at $z=0$ then the solution of the differential equation

$$\frac{dw}{ds}=I(w)$$

with boundary condition $w=z$ at $s=0$ is analytic both in s and z in a neighbourhood of $s=0$, $z=0$ and $w(s,z)$ satisfies $w\{s, w(t,z)\}=w(s+t,z)$. Thus regarded as a function of z with parameter s the function $f_s(z)=w(s,z)$, which has an expansion of the form (1), is a one parameter group embedding any of its members.

If we take, for example,

$$I(w)=\frac{w^2}{1-w},$$

the integrated form of the equation with the given boundary conditions is

$$w^{-1}+\log w=z^{-1}+\log z-s. \tag{12}$$

Putting $w=z(1+v)$ we obtain

$$z=\frac{v}{1+v}\{s+\log(1+v)\}^{-1}$$
$$=s^{-1}\{v-(s^{-1}+1)\,v^2+\cdots\}$$

regular in v at $v=0$ and this may be inverted to give

$$v=s\,z+(s+s^2)\,z^2+\cdots$$

regular in z at $z=0$. Thus

$$w=z+s\,z^2+(s+s^2)\,z^3+\cdots,$$

convergent for small enough s and z. Fix some sufficiently small non-zero s_0 and take

$$w=f(z)=z+s_0\,z^2+(s_0+s_0^2)\,z^3+\cdots.$$

This is then convergent in some neighbourhood of $z=0$ and is embeddable. In the notation of the theorem $f(z)$ is our series (1) and we have $a_2=s_0$, $a_3=s_0+s_0^2$ so that (b) is certainly fulfilled. It is clear that condition (a) cannot hold and indeed (12) shows that $f(z)$ is infinitely many-valued.

b) Embeddable Algebraic Functions

Take any algebraic function $A(t)$, for example $t+\beta t^{-1}$, which has a branch regular at ∞ for which

$$A(t)=t+\beta\,t^{-1}+\gamma\,t^{-2}+\cdots.$$

Then the inverse $A_{-1}(w)$ of $A(t)$ has an expansion

$$A_{-1}(w)=w-\beta\,w^{-1}-\gamma\,w^{-2}+\cdots$$

and the algebraic function

$$g(t)=A_{-1}\{A(t)+1\}=t+1+\beta\,t^{-2}+\cdots$$

is embeddable in the group of iterates

$$g_s(t)=A_{-1}\{A(t)+s\}=t+s+s\,\beta\,t^{-2}+\cdots.$$

One may transfer the fixpoint from ∞ to 0 without changing the nature of the function and obtains an embeddable algebraic

$$f(z)=\{g(z^{-1})\}^{-1}=z-z^2+z^3-(\beta+1)\,z^4+\cdots.$$

Thus there are indeed functions $f(z)$ satisfying condition (a) of the theorem but embeddable. Of course they must then satisfy $a_2^2-a_3=0$ if $a_2\neq 0$. In our example $a_2=-1$, $a_3=+1$.

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